THE BOOK WAS DRENCHED

PRACTICAL MATHEMATICS

PRACTICAL MATHEMATICS

INCLUDING TRIGONOMETRY AND AN INTRODUCTION TO THE CALCULUS

A Class-Book for Higher Elementary Secondary and Technical Schools

BY

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PREFACE

So many text-books on Algebra have appeared during the last few years that the publication of yet another requires some justification.

In the present volume, the processes of Algebra are established in a practical way, and their application to Science and Mensuration is made a prominent feature Although some boys find a certain satisfaction in juggling with meaningless symbols, the application of Algebra to practical problems rouses their interest and fills them with enthusiasm. The author finds encouragement in a recent address to the Mathematical Association by Professor Whitehead, F.R.S., which contained the following: "Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world."

Graphs are an important feature of this book, and they are not abandoned as soon as their algebraic significance becomes evident, but are employed to illustrate and establish many important relations. The treatment is in accord with the recent suggestions of the Board of Education.

Ratio and Proportion being of such importance in the teaching of Science, are given an early place in the scheme.

Trigonometry is introduced. Experience shows that when students recognise that trigonometrical terms are merely algebraic symbols, the terrors of the subject disappear.

Other special characteristics are :

I. Academic treatment has been avoided as far as possible.

II. The requirements of pupils who will engage in technical work on leaving school, but who will not have the opportunity to attend a University, have been borne in mind.

III. Artificial problems have here given place to more useful applications.

The calculus is included for its usefulness, and also with a view

to showing the pupil that a great realm lies beyond elementary Algebra.

Chapter XXV is added for those who wish to examine more closely the Binomial and Exponential Theorems, and the determination of logarithms. Some may prefer to take this chapter before Chapter XXII.

The exercises are of reasonable length, and in many cases are chosen with the idea of preparing the way to future work. The student should therefore work as many as possible.

No examination tests are included. A wise teacher prepares or selects these himself.

While the majority of the exercises are original, my acknowledgments are due to the Delegates of the Oxford Local Examinations, who have kindly allowed me to include several questions set at various examinations.

I have received valuable help from my staff, among whom should be mentioned by name, Mr. R. Hawksworth, B.Sc., Mr. S. Chatwin, B.Sc., and Mr. C. P. Le Huray, B.A., who have taken great interest in the work.

A. H. B.

August 1916.

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ALGEBRA

CHAPTER I

POSITIVE AND NEGATIVE NUMBERS, ZERO, SIGNS

EXAMPLES OF POSITIVE AND NEGATIVE NUMBERS

§ **1. Possessions and Debts.**

A man (A), after paying all his debts may find that he still possesses a sum of money or something equivalent to a sum of money. Another man (B) may find that, although able to discharge all his debts, he has nothing left. A third man (C) may find that, after paying off as many debts as he is able, he has still debts to pay.

If we agree to call the state of A **positive**, as is usual, then the state of C is **negative**, and that of B neither positive nor negative. Suppose that A's possession is £15, then in Algebra his state is shown as + £15 (spoken, **plus** fifteen). The mark + is called the **plus sign**, and + 15 is called a **positive** number.

If C's remaining debts amount to £15, his state is denoted by − £15 (spoken, **minus** fifteen). The mark − is called the **minus sign,** and − 15 is called a **negative** number. It will be observed that the state of C is precisely **opposite** to that of A.

The state of B is represented by the **zero** figure, 0. Notice that in the zero state, there is an absence of both possessions and debts.

Note.—When no sign is prefixed to a number, the number is positive.

EXERCISE I (A)

Answer the following:

1. After paying all debts, a man is worth £250. Represent his state algebraically, and contrast it with that of a man who owes £250.
2. After paying as many of his debts as possible, a man finds that he still owes £60. Represent his state algebraically.

3. Referring to Exercise 2, how much must the man earn before he is able to reach the zero state?
4. A man has £350, but his debts amount to £235. What is his actual state? Represent it algebraically.
5. Another man has £235, and his debts amount to £350. What is his actual state? Represent it algebraically.
6. A merchant, whose assets (possessions) amount to £350 and his debts to £200, joins in business with another merchant, whose assets amount to £500 and whose debts are £300. What are the assets and debts of the partnership? Represent its actual state algebraically.
7. A merchant, whose assets amount to £250 and whose debts are £350, joins in business with another merchant, whose assets are £350 and whose debts are £250. What is their joint state?
8. A merchant, whose assets amount to £250 and debts to £350, joins in business with another, whose assets are £150 and debts £200. What is their joint state?

§ 2. Thermometer Scales.

Examine carefully the scale of a Centigrade thermometer. If possible, obtain one thermometer in which the tube is vertical, as in fig. 1, and another in which it is horizontal, as in fig. 2.

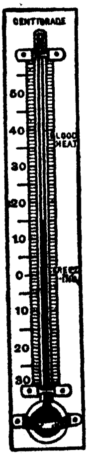


FIG. 1.

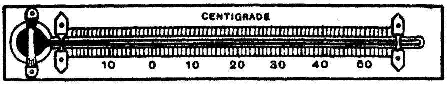


FIG. 2.

Observe that the divisions appear to be equal, and that they are numbered from a mark numbered 0. This is the **zero** mark.

In fig. 1, the numbers above the zero are said to be positive, and those below the zero, negative.

In fig. 2, the numbers to the right of the zero are positive, and those to the left, negative.

You probably know that the temperature is given by the position of the surface of the liquid in the thermometer, as shown by the scale. Thus in fig. 1, since the surface is opposite the 15 mark above the zero, the temperature is $+15°$ (the sign ° denotes degrees). In fig. 2, the surface is opposite the 10 mark to the left of the zero, and the temperature indicated is $-10°$.

EXERCISE I (B)

Answer the following questions referring to thermometer scales:

1. Represent algebraically the temperature when the surface of the liquid is:
   (*a*) 12 divisions above the zero.
   (*b*) 5 divisions below the zero.
   (*c*) at the zero.
   (*d*) 100 divisions above the zero.
   (*e*) 15 divisions below the zero.
2. The reading of a thermometer is $+10°$. If the temperature rises $15°$, what is the final reading?
3. The temperature falls from $+25°$ through $15°$. What is the final reading?
4. The temperature falls $25°$ from $+15°$. What is the final reading?
5. The temperature rises through $20°$ from $-5°$. What is the final reading?
6. The temperature is $-10°$. Through how many degree divisions must the surface rise in order to reach zero?
7. The zero of a thermometer is lowered through 5 degree divisions. What will be the new numbering of the graduations originally numbered, $+15$, -10, -5, -2, 0?
8. Water freezes at $32°$ F. and boils at $212°$ F. How many degrees are there between these temperatures?
9. Alcohol freezes at $-112°$ C. and boils at $78°$ C. How many degrees are there between these temperatures?

§ 3. Linear Measurements.

Draw a straight line BA (fig. 3), and in it mark a point O.

FIG. 3.

If measurements made in the direction OA are positive, then measurements made in the opposite direction, OB, are negative.

Notice that the direction from B to A is the same as the direction OA, and the direction from A to B as the direction OB.

EXERCISE I (c)

1. From a point Q, in a straight line, mark off the following distances in the directions indicated by the signs:

   + 3 inches, − 4 inches, − 6·2 inches, + 4·7 cms., − 5·8 cms.

2. From a point O, in a straight line, mark off OA equal to + 5·7 cms., then from A mark off a distance AB, − 3·4 cms. (fig. 4). Represent algebraically the distance OB.

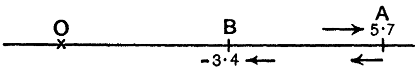


FIG. 4.

3. Draw a straight line, and in it mark a zero point. Measure off a distance OB, − 4·3 cms., and from B a distance BC, − 3·6 cms. What is the distance OC? (Remember the sign.)

4. From a point O, in a straight line, mark off OA, + 3·4 cms., and from A, mark off AB, − 6 cms. What is the distance OB? Why is OB negative?

§ 4. Angular Measurement.

Imagine the straight line OA (fig. 5) to be pivoted at O, and to be rotated without leaving the surface of the paper.

The line may be rotated either in the direction in which the hands of a clock move (clockwise), or in the contrary direction (counter-clockwise or anti-clockwise).

If counter-clockwise is taken to be positive rotation, as is usual, then clockwise is negative rotation.

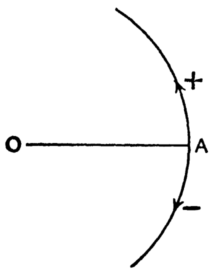


FIG. 5.

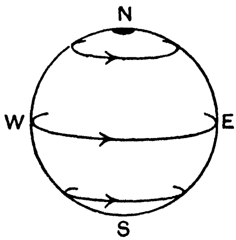


FIG. 6.

Looked at from the North Pole, the direction of rotation of the Earth is positive (see fig. 6).

Observe that the direction of the line OA changes as the line rotates, and is completely reversed when OA has rotated through

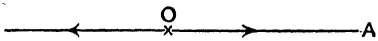


FIG. 7.

half a revolution in either the positive or negative direction (fig. 7).

EXERCISE I (D)

1. Mark a direction OA. Then, using a protractor, mark a direction OB $+35°$ from the direction OA, and another, OC, $-25°$ from OA. Measure the angle BOC.
2. From a standard direction OA, mark a direction OB, $+80°$. Now mark another direction, OC, $-60°$ from OB.

   How many degrees is the direction OC from the direction OA?
3. Draw the compass directions from a point O, as shown in fig. 8.

   From OE mark off an angle $+50°$; from OW, an angle $-35°$; from OS, an angle $-65°$; and from ON, an angle $125°$.

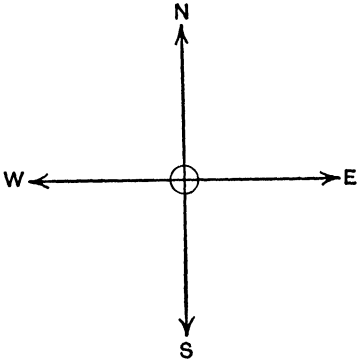


FIG. 8.

4. Sea level being the zero, express algebraically the height of a mountain 3028 feet high, and the depth of a mine 150 fathoms deep. (A fathom is 6 feet.)
5. Two equal toothed wheels are geared together. The first makes 60 revolutions per minute in the counter-clockwise direction. What is the direction of rotation of the second wheel? Represent the speeds algebraically.
6. Two pulleys are connected by a belt. The first makes 100 revolutions per minute in a clockwise direction, and the other twice as many revolutions per minute. Represent these speeds algebraically.
7. One clock (A) is 15 minutes fast, another (B) is 10 minutes slow. Represent the errors algebraically.

8. One clock (A) gains 8 minutes a day, another clock (B) loses 5 minutes a day. Represent these changes algebraically.

9. A body falling to earth gains speed at the rate of 32·2 ft. per second each second. A body projected upward from the earth loses speed at the rate of 32·2 ft. per second each second. Represent these changes in speed algebraically.

10. When a spring balance is pulled, the backward force of the spring is felt. If the reading is 12 lbs., represent algebraically the value of the pull, and that of the backward force of the spring.

11. A spiral spring can be stretched or compressed. How would you distinguish algebraically between these changes?

12. Represent 55 B.C. and 1915 A.D. algebraically.

CHAPTER II

SUM AND DIFFERENCE

§**1.** The pupil is warned that the conception of addition and subtraction usually formed from the rules of Arithmetic is not complete.

In Algebra, a much wider view of these processes is taken, and it will be seen that in finding the sum of numbers it is sometimes necessary to subtract the figures. It depends upon the sign of the numbers.

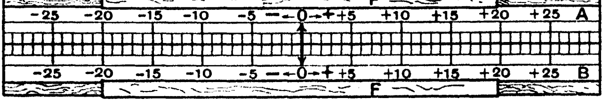


Fig. 1.

The teacher and pupils should construct in wood or cardboard the apparatus illustrated in fig. 1, which consists of two like scales with equal divisions numbered as shown.

One scale, A, is fixed and the other, B, movable. The two scales are contained in a frame F.

If the material mentioned is not available, the pupils should make a similar contrivance in their exercise books, the movable

scale being on a strip of paper passed through loops formed by making two parallel cuts through the page with a penknife.

On both scales, it will be seen that the positive numbers are on the right, and the negative numbers on the left of the zero mark.

With the help of these scales, it is proposed to establish some very important facts which should be remembered.

EXAMPLE i.—*Find the sum of* $+9$ *and* $+5$.

Find the $+9$ mark on the fixed scale, then bring the zero of the movable scale opposite this $+9$ mark, and find the number on the fixed scale to which the $+5$ on the movable scale is opposite. You will find it to be $+14$ (fig. 2).

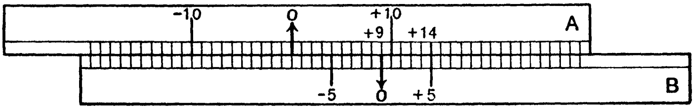


FIG. 2.

Of course you know this to be correct.

If this exercise had been done by drawing a line 9 cms. long, and then extending it by 5 cms., it would have been noticed that the second line (5 cms.) followed from the end of and in the same direction, namely the positive direction, as the first line (9 cms.).

In the following exercises, the second number is added to the first by proceeding from the end of the length representing the first, in the direction indicated by the sign of the second number.

EXAMPLE ii.—*Find the sum of* -9 *and* -5.

Find the -9 mark on the fixed scale; bring the zero of the movable scale to this mark, and find the number on the fixed scale to which the -5 on the movable scale is opposite.

You will find that the result is -14.

EXAMPLE iii.—*Find the sum of* $+9$ *and* -5.

Find the $+9$ mark on the fixed scale, then move the lower scale until the 0 is opposite this mark. The number on the fixed

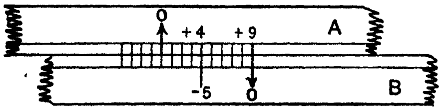


FIG. 3.

scale to which the -5 of the lower scale is opposite gives at once the result, namely $+4$ (fig. 3).

In such a case you see that, to find the sum, the numbers, regardless of sign, are actually subtracted.

Observe that the result is positive, because the number with the plus sign is greater than that with the minus sign.

Notice that the -5 *is measured in the direction opposite to that in which the* $+9$ *is measured.*

EXAMPLE iv.—*Find the sum of* $+5$ *and* -9.

Find the $+5$ mark on the fixed scale, move the lower scale until its zero mark is opposite the $+5$; then the number on the upper scale to which the -9 on the lower is opposite, is the sum of $+5$ and -9. The result is -4.

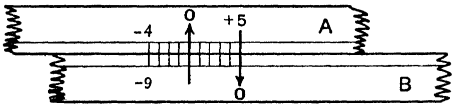


FIG. 4.

Observe that the result is negative, because the number with the minus sign is greater than the number with the plus sign (fig. 4).

Note.—The sign $+$ stands for addition also.

Thus $8+(-5)$ means, to $+8$ *add* -5.

The small bracket is used to separate the two signs.

EXERCISE II (A)

Using the scales as in the foregoing examples, find the sum of:

1. $+15$ and $+6$. 2. -3 and -8. 3. -8 and $+3$.

4. $+12$ and -7. 5. -4 and $+11$. 6. -1 and $+1$.

Without using the scales, write down the answers to the following:

7. Find the sum of $+8$ and $+5$, and of 4 and -9.

Add, that is, find the sum of,

8. -8 and -5. 9. $+8$ and -5. 10. -8 and $+5$.

You will now see the reason for the following rules for finding the *sum* of two algebraic numbers.

(1) If the *signs* of the numbers are *alike*, *add* the numbers and prefix the sign of the numbers.

(2) If the *signs* of the numbers are *unlike, subtract* the smaller number from the greater, regardless of signs, then prefix the *sign* of the *greater* number.

EXERCISE II (B)

Using the foregoing rules, find the sum of:

1. $+12$ and $+8$. 2. -12 and -8. 3. $+12$ and -8.
4. -12 and $+8$. 5. -20 and -5. 6. -20 and $+5$.
7. $+20$ and -5. 8. $+20$ and $+5$.
9. Using the scales, show that the result of adding $+8-5+3-5$ in order is the same as adding $+8+3-4$ and -5 in order, and as adding $+11$ and -9, i.e., the sum of the positive numbers and the sum of the negative numbers.

Find the value of the following:

10. $23-14+8-6-2+7$. 11. $-14+3-16-2+10$.
12. A ship sails E. for 2 hours at 12 miles per hour, turns and sails W. for $3\frac{1}{4}$ hours at 15 miles per hour, then turns again and sails E. for $1\frac{1}{4}$ hours at 10 miles per hour. How far, and in what direction, is it from the starting point?

§ 2. Difference.

Being asked the difference between 5 and 9, almost every boy would say at once, four. In Algebra, however, this answer is incomplete, because the difference depends upon whether we view the difference from the 5 or from the 9.

Work the exercise on the special apparatus.

The movable scale is now used to measure differences.

EXAMPLE. i.—Find the marks $+5$ and $+9$ on the fixed scale, move the lower scale until its 0 is opposite $+5$ on the upper scale; then the reading on the lower scale, which is opposite $+9$ on the upper, measures the difference between 5 and 9 when 5 is the number from which we reckon the difference. The answer is, of course, 4, meaning $+4$.

This exercise could have been stated as follows:

From 9 *subtract* 5. *Answer*, $+4$.

Or the exercise might have been stated as "*What does* 9 *become when* 5 *is made the zero?*"

Another way of stating the same exercise is, "*What must be added to* 5 *to make* 9?"

Measure the amount by placing the lower scale so that its zero is at 5.

It is seen that subtraction is the reverse process of addition.

Compare this result with that of Example iii., page 15.

You see that the result is the same as when 9 and -5 are added.

EXAMPLE ii.—Place the lower scale so that the zero is opposite the 9 of the upper scale, and see what reading on the lower scale is opposite the 5 of the upper scale (fig. 5).

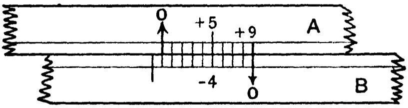


FIG. 5.

The answer is -4. That is, -4 is the difference between 9 and 5 when regarded from the 9.

The exercise might have been stated as follows:

(i) *From* 5 *subtract* 9. *Result*, -4.

(ii) *What must be added to* 9 *to give* 5? *Answer*, -4.

Compare this result with that of Example iv., page 16.

You see that the result is the same as when -9 and 5 are added.

EXAMPLE iii.—*From* -9 *subtract* -5.

In other words, find the difference between -9 and -5 regarded from -5.

Find these positions on the upper scale.

Move the lower scale until the zero is at -5, and measure the difference. *Answer*, -4.

Comparing with Example iv., page 16, it is seen that the result is the same as that obtained by adding $+5$ and -9.

EXAMPLE iv.—*From* -5 *subtract* -9.

Placing the zero of the lower scale at -9 of the upper, the result is seen to be $+4$.

EXAMPLE v.—*From* -9 *subtract* $+5$.

Proceed as before. See that the zero of the lower scale is at $+5$. The answer is -14, which is the same as that obtained by adding -5 to -9.

EXAMPLE vi.—*From* $+5$ *subtract* -9.

Placing the zero of the lower scale at -9, the result is found to be $+14$ (fig. 6).

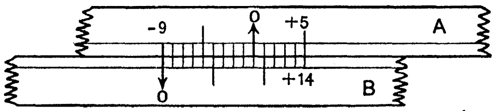


FIG. 6.

Summing up these exercises, it is seen that:

(i) *Subtracting a positive number is equivalent to adding a negative number.*

(ii) *Subtracting a negative number is equivalent to adding a positive number.*

The rule for subtraction is: *Change the sign of the number to be subtracted, and then proceed as in finding the sum.*

The following examples are interesting, because they illustrate this change in sign.

EXAMPLE i.—*From* 0 *subtract* 5.

Place the zero of the lower scale at 5, and find to what reading on the lower scale the 0 of the upper is opposite. *Answer*, -5.

In other words, -5 must be added to $+5$ to give 0; or, if the zero is moved to 5, the original 0 becomes -5.

EXAMPLE ii.—*From* 0 *subtract* -5.

Proceeding as before, the answer will be found to be $+5$.

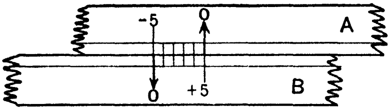


FIG. 7.

You will notice that moving the zero to the -5 mark reverses the direction of the length from 0 to -5 (fig. 7).

Note.—The minus sign is the sign for subtraction also.

Thus, $9-(-5)$ means, "From 9 subtract -5."

We have seen that this is equal to $9+5$.

Exercise II (c)

Verify each result by working the exercise on the special apparatus.

Subtract:

1. 6 from 10.   2. -6 from -10.   3. -10 from 6.

4. 10 from -6.   5. -10 from -6.   6. 10 from 6.

7. -6 from 10.   8. 6 from -10.

By how much does:

9. -18 differ from 10?   10. -18 from -10?

11. 18 from 10?   12. 18 from -10?

Subtract:

13. 5 from 0.   14. -5 from -5.   15. -3 from -7.

16. 5 from -5.   17. -5 from 5.   18. -7 from 3.

Find the value of the following:

19. $9-(-4)$.   20. $4-(-9)$.   21. $-3-(-8)-2$.

22. $0-(-5)$.   23. $0-(+5)$.   24. $9-(2-7)$.

25. $6-2+4-(5+3-1)$.   26. $6-(4+2-9)$.

27. $32-15-7+6-(-4)$.   28. $-(3-6+5)-(-2+8-4)$.

CHAPTER III

MULTIPLICATION AND DIVISION OF POSITIVE AND NEGATIVE NUMBERS

§1. Multiplication.

Example i.— $+4$ *multiplied by* $+2$.

This may be taken to mean, on our scale, two consecutive lengths, each $+4$.

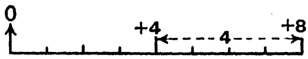


Fig. 1.

That is, a length 8 (fig. 1).

It is also a short way of writing two fours added, i.e. $4+4$.

It will be readily understood that the sum of any number of positive numbers is positive. Thus, 5 fours added give $+20$.

The product of two positive numbers is positive.

EXAMPLE ii.— -4 *multiplied by* $+2$.

This may be taken to mean two consecutive lengths, each -4. That is, a length -8 (fig. 2).

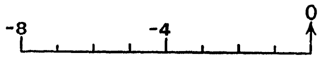


FIG. 2.

It is also a short way of writing two minus fours added, i.e. $-4+(-4)=-4-4=-8$.

EXAMPLE iii.— $+4$ *multiplied by* -2.

This may be taken to mean four minus twos added, i.e.

$$-2+(-2)+(-2)+(-2)=-8.$$

Or that twice four has to be a subtracted number. Thus, from 0 subtract twice four.

$$0-2\times 4 \text{ equals } 0-8, \text{ i.e. } -8.$$

The product of a positive and a negative number is negative.

The operations 4×2 and -4×2 are contrasted graphically in fig. 3, in which AC is 4, AB 4×2, and AD -4×2.

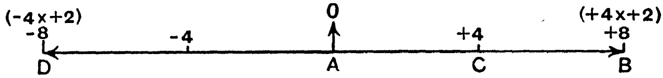


FIG. 3.

Observe that 4×2 is changed into -4×2 by turning AB round A until its direction is reversed.

EXAMPLE iv.— -4 *multiplied by* -2.

This may be taken to mean that twice minus four has to be a subtracted number. Now twice minus four is minus eight, and subtracting minus eight is equivalent to adding plus eight.

As an example of this, consider the following:

From 0 *subtract twice minus* 4.

$$0-(-8)=0+8=+8.$$

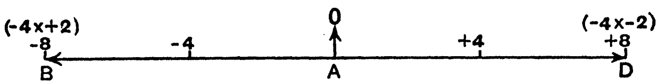


FIG. 4.

If AB (fig. 4) represents -4×2, then -4×-2 is obtained by

turning AB about A until its direction is reversed, i.e. until it is positive.

The product of two negative numbers is positive.

§ **2.** These examples, being fundamental, are so important that we shall illustrate them in another manner.

A product can be represented by the area of a rectangle or oblong. You probably know that the area of such a figure is found by multiplying the length by the breadth.

EXAMPLE i.— $+4 \times +2$.

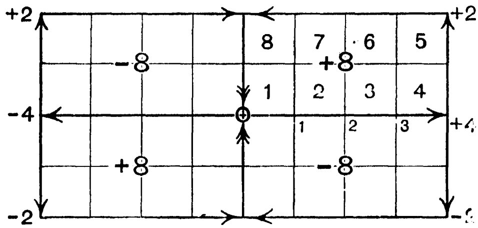


FIG. 5.

If we mark out the length $+4$ in a horizontal direction to the right, and from the end draw in the upward vertical direction the breadth $+2$, as shown in the figure, we turn in the anti-clockwise or positive direction of rotation.

The boundaries of the rectangle can be followed round in the anti-clockwise direction. We can agree to call an area, enclosed by boundaries described in this way, positive,* and so illustrate that the product of $+4$ and $+2$ is $+8$.

EXAMPLE ii.— $-4 \times +2$.

The boundaries of the rectangle in this case are described in the *clockwise* or *negative* direction. The result is therefore -8 (fig. 5).

EXAMPLE iii.— $+4 \times -2$.

The figure shows the result to be -8.

EXAMPLE iv.— -4×-2.

The figure shows that since the area is described in the *anti-clockwise* or *positive* direction, the result is $+8$.

* This convention is used in Engineering.

§ **3.** Summing up these results, we have:

The product of *two positive* numbers is *positive.*

The product of a *positive* and a *negative* number is *negative.*

The product of *two negative* numbers is *positive.*

This is often stated briefly in the form:

Two like signs give plus; two unlike signs, minus.

In signs:

$+$ multiplied by $+ = +$, $\quad + $ multiplied by $- = -$,

$-$ multiplied by $- = +$, $\quad -$ multiplied by $+ = -$.

EXERCISE III (A)

1. 6×3. 2. 6×-3. 3. -6×3 4. -6×-3.
5. 1×1. 6. -1×1. 7. 1×-1. 8. -1×-1.
9. $\frac{3}{4} \times -\frac{1}{2}$. 10. $-1\frac{1}{2} \times -\frac{2}{3}$. 11. 0×3. 12. 0×0.
13. -2×0. 14. $-3{\cdot}2 \times 5$. 15. $3 \times -6 \times 5$.
16. $4 \times -5 \times -3$. 17. $-3 \times -6 \times -2$. 18. -2×-2.
19. $-2 \times -2 \times -2$. 20. $5 \times -3 \times -6 \times 2 \times -4\frac{1}{2}$.
21. A clock gains 5 minutes a day. If it is correct now, how much will it be wrong 3 days hence, and how much was it wrong 3 days ago? Another clock loses 5 minutes a day. If it is correct now, how much will it be wrong 3 days hence, and how much was it wrong 3 days ago?

    Select signs for gain and for loss, and for fast and for slow, and represent these four results algebraically.
22. What is the sign of the product of:

    (i) An odd number of negative numbers?

    (ii) An even number of negative numbers?

§ **4. Division.**

Division is the inverse of multiplication.

EXAMPLES.

i. Since $+4$ multiplied by $+2$ gives $+8$,

therefore $+8$ divided by $+2$ gives $+4$,

and $+8$ divided by $+4$ gives $+2$.

ii. Since -4 multiplied by $+2$ gives -8,

therefore -8 divided by $+2$ gives -4,

and -8 divided by -4 gives $+2$.

iii. Since -4 multiplied by -2 gives $+8$,
therefore $+8$ divided by -2 gives -4,
and $+8$ divided by -4 gives -2.

We learn from:

i. That a *positive* number divided by a *positive* number gives a *positive* number.

ii. That a *negative* number divided by a *positive* number gives a *negative* number, and that a *negative* number divided by a *negative* number gives a *positive* number.

iii. That a *positive* number divided by a *negative* number gives a *negative* number.

In signs:

$+$ divided by $+ = +$, $\quad$ $-$ divided by $+ = -$,
$-$ divided by $- = +$, $\quad$ $+$ divided by $- = -$.

§ **5.** The rule is seen to be the same as that for multiplication, namely, "*Two like signs give plus; two unlike signs give minus.*"

§ **6. Miscellaneous Examples.**

Note.—Except where indicated by brackets, multiplication and division must be done before addition and subtraction.

EXAMPLE i.—Simplify, $6+2\times -3-6\times -4+7$.

$$\begin{aligned} 6+2\times -3-6\times -4+7 &= 6-6+24+7 \\ &= 31. \end{aligned}$$

EXAMPLE ii.—Simplify, $(6+2)\times -3-6\times -4+7$.

$$\begin{aligned} (6+2)\times -3-6\times -4+7 &= 8\times -3+24+7 \\ &= -24+24+7 \\ &= 7. \end{aligned}$$

EXERCISE III (B)

Divide

1. 15 by 3. 2. 15 by -3. 3. -15 by 3. 4. -15 by -3.
5. 3 by 15. 6. 3 by -15. 7. -3 by 15. 8. -3 by -15.

Find.

9. $\dfrac{18}{-3}$. 10. $\dfrac{-18}{-3}$. 11. $\dfrac{-3}{-1}$. 12. $\dfrac{6}{-2}$.

13. $\dfrac{1}{2}$ of -4. 14. $\dfrac{0}{4}$. 15. $\dfrac{-20}{-4}$.

16. $3 \times -2 + 9 - 4 \div -2$.

17. $(2-3) \times -6 - (4-2) \times 3$.

18. $-3 \times 4 - 6 \times -2 + \dfrac{15}{-3}$.

19. $\dfrac{8-2}{2} + 5 \times -2$.

20. $-\dfrac{4-8}{2} + \dfrac{10-2}{3+1}$.

21. $(8-6) \times (6-8)$.

22. $(5-9) \times -3$.

23. Verify that $(5-3)$ multiplied by $(4-7)$ is equal to

$$(5-3) \times 4 + (5-3) \times -7,$$

and is equal to $\quad 5 \times 4 - 3 \times 4 + 5 \times -7 - 3 \times -7$.

24. Simplify i. $\dfrac{\dfrac{-6-9}{4}}{\dfrac{25-20}{-8}}$. ii. $\dfrac{-\dfrac{6-9}{4}}{\dfrac{25-20}{-8}}$.

CHAPTER IV

SYMBOLS, COEFFICIENTS, COMMON PROCESSES WITH SYMBOLS

§**1.** In Algebra numbers are often represented by letters.

Draw a straight line AB, of any length, and another CD (fig. 1).

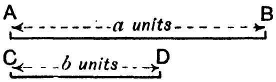


FIG. 1.

It is possible to perform certain operations with these straight lines without knowing their length.

Let us represent the number of units of length in AB by the letter a, and the number of units in CD by the letter b; then, if

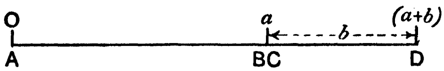


FIG. 2.

we place AB and CD together in one straight line as in fig. 2, the length of the whole line AD is the sum of a units and b units, which is written $(a+b)$ units.

No simpler answer than $(a+b)$ is possible, since we do not know the value of a and b.

Of course, if we know a to be 3 inches and b to be 2 inches, then we can say that $(a+b)$ is 5 inches.

Letters or other characters used to denote numbers are called **Symbols:**

§ **2.** An arrangement of symbols, such as $a+b$, is called an **Expression.**

§ **3.** The parts of an expression connected by plus or minus signs are called **Terms.** It will be seen later that a term may consist of many symbols.

§ **4.** When an expression is used to denote a single measurement like the length of one line, it is a good plan to enclose it in brackets, for the expression acts as one symbol.

Thus the length AD (fig. 2) is $(a+b)$, but it might have been denoted by a single letter, say x.

§ **5.** The sum of a and b, then, is $a+b$.

Similarly, the sum of x and y is $x+y$; of x and 2, $x+2$; and so on.

§ **6.** If CD is subtracted from AB, as shown in fig. 3, the number of units of length in AD is $(a-b)$, and this is the simplest statement for the result of subtracting b from a.

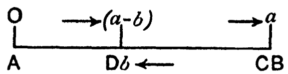


FIG. 3.

As before, a simpler result could be obtained if we knew the value of a, and of b.

Hence, $\quad b$ subtracted from $a = a - b$.

Similarly, $\quad a$ subtracted from $b = b - a$,

y subtracted from $x = x - y$,

x subtracted from $y = y - x$,

2 subtracted from $x = x - 2$, and so on.

§ **7.** Now suppose that the lines AB and CD happen to be equal in length. Then, if AB measures a units, CD also will measure

a units, and on adding AB and CD together, we shall obtain a line AD of $(a+a)$ units (fig. 4).

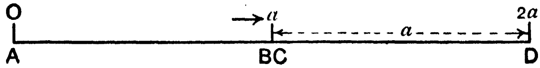


Fig. 4.

Now $(a+a)$ is written more simply as $2a$.

If another equal length, DE, is added, then AE is $(a+a+a)$ or $3a$ units.

Similarly, five a's added, i.e. $(a+a+a+a+a)$ are written shortly as $5a$.

§ **8.** The number 5 in $5a$ is called the **coefficient** of the term.

The coefficient indicates **the number of the symbols** (or groups of symbols) *added* together.

Note.—The term $5a$ may be taken to mean also 5 times a, although no multiplication sign is placed between the figure and the symbol.

Remember, then, that such a term as $7x$ is a short way of writing seven x's added,

i.e. $7x = x+x+x+x+x+x+x.$

§ **9. Examples in Addition.**

(i) *Add* $3a$ *to* $5a$.

$5a$ means $a+a+a+a+a.$

$3a$ means $a+a+a.$

Hence $5a+3a = a+a+a+a+a+a+a+a,$

i.e. eight a's added, which is written $8a$.

$$\therefore\ 5a+3a=8a.$$

Notice that the coefficient of the sum is obtained by adding (algebraically) the coefficients (5 and 3) of the terms.

This applies only to like terms.

(ii) *Find the sum of* $-3a$ *and* $5a$.

$$\begin{aligned} -3a+5a &= (-3+5)a \\ &= 2a. \end{aligned}$$

(iii) *Find the sum of* $-5a$ *and* $3a$.

$$\begin{aligned} -5a+3a &= (-5+3)a \\ &= -2a. \end{aligned}$$

(iv) *Find the sum of* $6a$, $-9b$, $-2a$, $+4b$ *and* $-a$.

Sum of the terms containing $a = 6a - 2a - a = 3a$.

Sum of the terms containing $b = -9b + 4b = -5b$.

Sum of all the terms $= 3a - 5b$.

EXERCISE IV (A)

Simplify the following:

1. $2a + 5a$.
2. $5x + 3x$.
3. $x + 2x$.
4. $3a + 2a + 4a$.
5. $2y + y + 5y$.
6. $2a + 6c + 2c + c$.
7. $6x + 2x - x$.
8. $-6x + 2x + 5$.
9. $5p + 3p$.
10. $5p + (-3p)$.
11. $-5p + (-3p)$.
12. $3a - 2a + 5b - 2b$.

Write down algebraically:

13. $3a$ and $2b$ added.
14. $-5x$ and $2y$ added.
15. $5x$ and $-2y$ added.
16. $5x$ and $-2x$ added.
17. A vessel when empty weighs w grams. If x grams of water are poured in, what is the total weight?
18. What is the total weight of the solution formed by dissolving x grams of salt in 100 c.c. of water? (1 c.c. of water weighs 1 gram.)
19. Draw a straight line 1·3 inches long and another 1·6 cms. long. Call the length of the first x units, and that of the second y units. Now draw a line $(3x + 2y)$ units long, and another representing the sum of $-3x$ and $-2y$ units.

    Measure all the lines in cms., and check your results.
20. Collect like terms, and so simplify the following:
$$3a + 2b + 4a + 5b + a + b + c + 3b + 5c.$$
21. Simplify
$$3a - 2b - 4a + 5b + a - b - c - 3b + 5c.$$
22. Simplify $\quad 3x + 4a - b - x + 2b - 3a + 4b,$

    and find its value when $x = 2$, $a = -1$ and $b = 3$.
23. Add the columns of the following example:

$$\begin{array}{r} a + 2b + 3c \\ 2a + \ \ b - 2c \\ -5a - 4b + \ \ c \\ 3b - 2c \\ \hline 6a - 2b - 3c \\ \hline \end{array}$$

Find the value of each line and of the sum when $a=1$, $b=2$ and $c=3$, and thus check your result. Why would the value $a=0$ not check the sum of the first column?

24. Arrange the following expressions as in Exercise 23, and find their sum:

$3a - 2x + y,\qquad 2x - 3y,\qquad -2a + 5x - 6y,$

$8a + 3y,\qquad -5a + x - 5y,\qquad 4a + x + 2y.$

Choose suitable values for these symbols, and check your result.

25. When $x = 2$ and $y = -3$, find the value of each of the following:

(i) $3x + 2y$. (ii) $3x - 2y$. (iii) $2x + 3y$.

(iv) $-2x + 3y$. (v) $\frac{3x}{2} + \frac{2y}{3}$. (vi) $\frac{3x}{2} - \frac{2y}{3}$.

(vii) $3a + 2x - 3y$. (viii) $\frac{x}{3} + \frac{3y}{4} + c$.

26. When $a = 0$, $b = -2$ and $c = 1$, find the value of:

(i) $3a - 2b + 3c$. (ii) $\frac{2a}{3} + \frac{3b}{4} - \frac{3c}{2}$.

27. What do the following become when a, b and c are all equal?

(i) $2a - 3b + c$. (ii) $7a + 2b - 3c$. (iii) $\frac{3a}{5} - b + 2c$.

§ 10. Difference.

From Example i. page 27, it will be readily understood that when $3a$ is subtracted from $8a$ the result is $5a$,

$$\text{i.e. } 8a - 3a = 5a.$$

The coefficient 5 is obtained by subtracting the coefficient, 3, of the term to be subtracted from the coefficient, 8, of the other term.

The process may appear simpler when represented as follows:

$$8a - 3a = (8 - 3)a = 5a.$$

Observe that the terms are alike.

Similarly, (i) $4a - 7a = -3a$.

(ii) $-4a - 7a = -11a$.

(iii) $-4a - (-7a) = -4a + 7a$

$= 3a$.

§ **11.** When subtracting one expression from another, it is sometimes more convenient (though not often) to use the arithmetical arrangement, thus:

From $3a+2b-3c$ take $2a-7b-c+d$.

$$\begin{array}{ll} \text{From} & 3a+2b-3c \\ \text{Subtract} & \underline{2a-7b-c+d} \\ & a+9b-2c-d \quad \textit{Answer}. \end{array}$$

By the rule for subtraction, page 19, the above example is equivalent to:

$$\begin{array}{ll} \text{To} & 3a+2b-3c \\ \text{Add} & \underline{-2a+7b+c-d} \\ & a+9b-2c-d \quad \textit{Answer}. \end{array}$$

Exercise IV (b)

1. (i) From $6a$ take $2a$. (ii) From $2a$ take $6a$.
(iii) From $6a$ take $-2a$. (iv) From $-6a$ take $2a$.
(v) From $-6a$ take $-2a$. (vi) From $-2a$ take $-6a$.
(vii) From $2a$ take $-6a$. (viii) From $-2a$ take $6a$.

Check each result by working the reverse operation with it and the term subtracted.

2. Using the same values for x and y as in No. 19, Ex. IV (a), find straight lines to represent:

(i) $(3x-2y)$. (ii) $(3x-4y)$. (iii) $(x-3y)$.
(iv) $(2x-5y)$. (v) $(-x+3y)$. (vi) $(-x-y)$.
(vii) $(-2x-(-3y))$. (viii) $0-(-2y)$.

In each case say whether the result is positive or negative.

Check your drawing by measurement and calculation.

3. From $3a-2b+7c$ subtract $2a+5b-c$.

4. (i) From $2x$ subtract 2. (ii) From $2x$ subtract x.
(iii) From 0 subtract x. (iv) From x subtract $-x$.

5. From $2a-5b+3$ take $3a+2b+5$.

Check your result by substituting 1 for a and 2 for b in each given expression and in the answer.

6. Simplify

$3a - 2b + 7c + 2a + 5b - c + 3b - 4c - 5a + 3b - 5c - (2a + 2b + c)$.

Find the value when $a = -2$, $b = 3$ and $c = -1$.

7. Find the difference between:

(i) $-x$ and x. (ii) 0 and a.

(iii) 3 and $3x$. (iv) $-x$ and y.

8. An empty crucible weighs x grams. When some copper filings are placed in it, the whole weighs y grams. After heating, the crucible and contents weigh z grams.

Find (i) The weight of copper filings taken.

(ii) The gain in weight after heating.

9. A test-tube and its fittings weigh w grams.

m grams of pyrolusite and n grams of chlorate of potash are placed in the tube and the whole heated.

If the weight is then x grams, what is the loss in weight?

If only the chlorate loses weight, what is the loss?

10. A copper ball weighs C grams in air, W grams in water and M grams in turpentine.

Find

(i) The loss of weight in water.

(ii) The loss of weight in turpentine.

(iii) The excess of the loss of weight in water over the loss of weight in turpentine.

§ 12. Product.

We have seen that $3x$ may be taken to mean three times x, or three x's added.

If we were not aware of the number of x's, we could denote it by another symbol, say a, and write ax for the result.

This result ax may be taken to mean also a times x.

Notice that there is no sign between the two symbols.

Similarly, we have:

$x \times y = xy$, $\quad x \times -y = -xy$, $\quad -x \times y = -xy$,

$-x \times -y = xy$, $\quad a \times b = ab$, $\quad a \times -b = -ab$,

$3a \times 2b = 6ab$, $\quad a \times b \times c = abc$, $\quad 3a \times b \times 2c = 6abc$.

It should be noted that of the product $6abc$, 2, 3, a, b *and* c *are called* **factors.**

Fig. 5 represents graphically the products of:
(i) a and b, (ii) a and $-b$, (iii) $-a$ and b, (iv) $-a$ and $-b$.

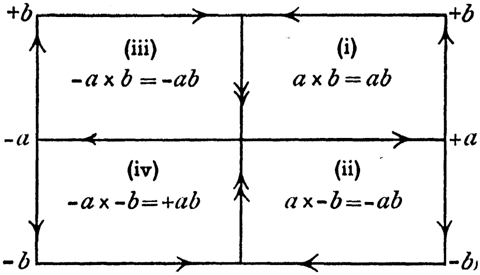


FIG. 5.

§ 13. Powers and Indices.

We have seen that: $x \times y = xy$.

Suppose now that x and y happen to be equal; then, if we represent these numbers by straight lines, the lines will be equal, and the rectangle which represents the product a square of area x times x (fig. 6).

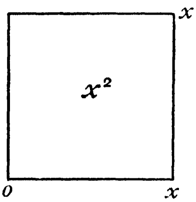


FIG. 6.

This product is not written xx, but x^2. (Spoken, x squared.)

The figure 2 is called an **index** (plural indices).

It indicates the number of x's *multiplied* together.

The number x^2 is called also the *second* **power** of x.

Similarly $x^3 = x \times x \times x$ (the third power of x).
$x^6 = x \times x \times x \times x \times x \times x$ (the sixth power of x).

Remembering the rule of signs,

$$-x \times x = -x^2, \qquad -x \times -x = +x^2,$$
$$-x^2 \times y = -x^2y, \qquad -x^2 \times -y = +x^2y,$$
$$-x \times x \times x = -x^3 \text{ (for } -x \times x = -x^2 \text{ and } -x^2 \times x = -x^3).$$

§ 14. Products of Powers.

To find $x^3 \times x^2$.

x^3 means three x's multiplied together,
and x^2 means two x's multiplied together;

then, $$x^3 \times x^2 = \underbrace{x \times x \times x} \times \underbrace{x \times x},$$

i.e. (three plus two) x's multiplied together, i.e. x^5.

Hence $$x^3 \times x^2 = x^{3+2} = x^5.$$

N.B.—$x^3 \times x^2$ is not x^6, but x^5.

In words, *the index of the power obtained by multiplying given powers of the same symbol is the* **sum** *of the indices of the given powers.*

EXAMPLES.
$$x^3 \times x^4 = x^{3+4} = x^7,$$
$$x^3 \times -x^4 = -x^{3+4} = -x^7,$$
$$-x^3 \times -x^4 = +x^{3+4} = x^7,$$
$$x^3y \times x^2 = x \times x \times x \times y \times x \times x$$
$$= x \times x \times x \times x \times x \times y$$
$$= x^5y,$$
$$x^2y^3 \times -x^3y = -x^{(2+3)}y^{(3+1)}$$
$$= -x^5y^4,$$
$$(x^2)^3, \text{ i.e. } x^2 \times x^2 \times x^2 = x^{2+2+2} \text{ or } x^{3 \text{ times } 2} = x^6.$$

Note.—When x^2 and x are added, the result is $x^2 + x$. The sum cannot be stated as a single term. Terms which contain the same symbols to the same power are called **like** terms.

EXERCISE IV (C)

Complete the following:

1. $x \times a =$ 2. $-x \times -a =$ 3. $ab \times x =$

4. $-3a \times 2x =$ 5. $-\frac{2}{3}x \times 3y =$

6. By means of a figure, show that $(x+2) \times 3$ equals $3x+6$.

7. Multiply $x+2$ by a. 8. Multiply $x+2$ by -3.

9. Multiply $x+2$ by $3a$ and by $-3a$.

10. Multiply $3a - 2b$ by $-2x$. Verify your answer by giving numerical values to a, b and x.

11. Taking straight lines 1·3 inches long and 1·6 cms. long respectively to represent x and y, construct figures to represent

(i) $6xy$. (ii) $x(x+y)$. (iii) $x(2x-y)$.

(iv) $2x(3x-2y)$. (v) $x(x+2'')$. (vi) $x(3x-2'')$.

12. Write down answers to the following:

$$a \times a, \quad a \times -a, \quad -a \times a, \quad -a \times -a.$$

13. Show by a drawing that $x \times 3x = 3x^2$, and that

$$x(3x+y) = 3x^2 + xy.$$

14. Write down the answers to the following:

(i) $2x^2y^2 \times -3xy$. (ii) $3ab \times b^2$. (iii) $(2x)^2$.
(iv) $-(2x)^2$. (v) $(-2x)^2$. (vi) $(x^3)^2$.
(vii) $(-3a^2b)^2$. (viii) $(-a)^3$. (ix) $(2a^2)^3$.

Find the value of each of the above when $x = 2$, $y = 3$, $a = -3$ and $b = -1$.

15. The area of a triangle is half that of the rectangle having the same base and altitude (height). If the base is b and the altitude h the area is $\frac{1}{2}bh$.

Find the areas of triangles having the following dimensions:

Base.	Altitude.
x	y
$2x$	y
y	$2x$
$3b$	$3h$
b	6

§ 15. Division.

The value of a divided by b is written as $\frac{a}{b}$.

If the terms of the quotient are powers of the same symbol, e.g. $\frac{a^5}{a^3}$, the answer can be simplified by cancelling.

Thus,
$$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^2,$$

i.e.
$$\frac{a^5}{a^3} \quad \text{or} \quad a^5 \div a^3 = a^{(5-3)} = a^2.$$

Notice that the index of the quotient is obtained by *subtracting* the index of the divisor from the index of the term divided. (*This is of course the reverse of the rule for multiplication, given on* page 33.)

Thus,
$$a^4 \div a^3 = a^{4-3} = a^1 = a.$$

Similarly,
$$\frac{x^7}{x^4} = x^{7-4} = x^3$$

and $$\frac{-30x^7}{6x^4} = -5x^3$$

and $$\frac{x^2y^3z}{xy} = xy^2z.$$

It should be borne in mind that a product is always exactly divisible by each of its factors, and that the quotient in each case is the product of the other factors.

Exercise IV (d)

Simplify :

1. $\frac{-a^2}{a}$. 2. $\frac{-a^2}{-a}$. 3. $\frac{9x^5}{3x^2}$. 4. $-\frac{2x^3}{6x^7}$.

Divide :

5. $-12a^3$ by $4a$.
6. $-3b^5$ by $-2b^2$.
7. $12a^3b^2$ by $4ab$.
8. $-8x^2y$ by $2x$.
9. $3a^2+12ab$ by $3a$.
10. x^2-xy^2 by x.

Check each answer by working the reverse operation.

Simplify :

11. $\frac{5xy \times 2ax}{axy}$. 12. $\frac{2ab \times -3ac}{12bc}$. 13. $\frac{x^2+xy^2}{2x}$.

14. $\frac{x^2y-3x^2y^3}{xy}$. 15. $\frac{-15a^3b^3-3ab^2+12ab}{-3ab}$.

16. Add
$$\begin{array}{r} 3x^2-2xy+\ y^2 \\ -x^2+5xy-6y^2 \\ 2x^2 \qquad\ \ -4y^2 \\ -4x^2-3xy+5y^2 \\ \hline \end{array}$$

17. From $3x^2+2x+1$ subtract $2x^2-3x+5$.

18. Multiply $\frac{2}{y}$ by 3.

19. Add $\frac{3}{y}$ and $\frac{2}{y}$. Verify your answer by putting y equal to 4.

20. From $\frac{9}{x}$ subtract $\frac{3}{x}$. Verify your answer.

21. When $x=2$ and $y=-3$, find the value of :

(i) $\frac{x^2}{y}$. (ii) $\frac{-3x^2}{2y^2}$. (iii) $-\frac{3x^2y^3}{xy}$.

(iv) $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{y}{x}$. (v) $\frac{xy}{x+y}$. (vi) $\frac{6}{x}-\frac{9}{y}$.

22. Divide the numerator and the denominator of $\frac{x^2}{x^5}$ by x^2.

23. Divide the numerator and the denominator of $\frac{ax^3 - bx^3}{ax^3 + bx^3}$ by x^3, and giving numerical values to a, b and x. Show that the value is not altered.

24. Simplify : $\frac{(8)^2 \times 3}{4} + \frac{(15)^2 \times 4}{5}$.

§ 16. Special cases of Multiplication.

Square and Square Root.

We have seen that $x \times x = x^2$.

This operation is called "Squaring." Squaring consists of multiplying a number by the same number.

Thus, the square of $-3a$ is $-3a \times -3a = 9a^2$.

§ 17. The reverse operation consists of finding what number must be squared to obtain a given number. The result is called the *square root* of the given number.

The sign for the operation is $\sqrt{\ }$.

Thus, $\sqrt{x^2}$ is x, because $x \times x$ is x^2.

This statement is not sufficient, for $-x \times -x$ is x^2.

There are therefore two square roots of x^2, namely, $+x$ and $-x$. These roots are usually written together in the form $\pm x$. (Spoken, plus or minus x.)

There are two square roots of every positive number.

EXAMPLES.

$$\sqrt{9} = \pm 3, \text{ because } 3 \times 3 = 9 \text{ and } -3 \times -3 = 9.$$

$$\sqrt{9x^2} = \pm 3x, \quad \sqrt{x^6} = \pm x^3, \quad \sqrt{16x^8y^6} = \pm 4x^4y^3.$$

Notice that in finding the *square* of a term, the index is *doubled*; and in finding the *square root*, the index is *halved*.

The square root of x is usually written $\sqrt{x}$, but may be written $\pm x^{\frac{1}{2}}$.

§ 18. Surds.

Surds are roots which cannot be written in terminating form. For example, $\sqrt{8}$ is not quite 3, but 2 decimal something. The decimal part neither recurs nor terminates. However, since $8 = 4 \times 2$ and $\sqrt{4}$ is 2, we can write $\sqrt{8}$ more simply as $2\sqrt{2}$.

(Observe carefully that $2\sqrt{2}$ means twice $\sqrt{2}$ and not $2+\sqrt{2}$. Contrast this with $2\frac{1}{2}$, which means $2+\frac{1}{2}$.)

Similarly, $\sqrt{18}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$,

$\sqrt{12}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$.

Surds can be treated exactly as algebraic symbols.

§ 19. Logarithms.

Another name for index is Logarithm (abbreviation, log).

Thus in a^3, 3 is the logarithm, and a is called the **base**.

These facts are usually expressed as follows:

The logarithm of a^3 to the base a is 3.

The statement means that 3 is the index of the power to which the base a must be raised to give the answer a^3.

$10^3=1000$, therefore the log of 1000 to the base of 10 is 3,

$10^5=100000$, „ „ 100000 „ 10 is 5.

The short way of writing these is

$$\log_{10}1000=3,$$

$$\log_{10}100000=5.$$

Similarly, $\log_a(a^3)=3$.

Notice that the base is written to the right of, lower down and smaller than the abbreviation log.

Exercise IV (e)

1. Find the square of:

   1, -1, $2x$, $-3x$, $-5x^2$, $3x^3$, $4\sqrt{x}$, $\sqrt{-x}$, $\sqrt{x^3}$, $3(a+b)$.

2. Find the square root of the following. Verify your answers by squaring:

   $25a^2$, $9b^4$, $49b^6$, $64y^{10}$, $25(a+b)^2$.

3. Why cannot you find the square root of -9?

4. Simplify:

   $$\sqrt{\frac{x^2}{4y^4}},\quad \frac{\sqrt{x^2}}{4y^4},\quad \sqrt{\frac{1}{x^2}},\quad \frac{1}{\sqrt{x^2}},\quad \sqrt{x^2+8x^2}.$$

5. Write as simply as possible:

$\sqrt{16}$, $\sqrt{25}$, $\sqrt{64}$, $\sqrt{100}$, $\sqrt{144}$, $\sqrt{12}$, $\sqrt{20}$, $\sqrt{28}$, $\sqrt{72}$, $\sqrt{32}$, $-\sqrt{75}$, $3\sqrt{16}$, $3\sqrt{32}$, $x\sqrt{8}$, $\sqrt{8x^2}$, $\sqrt{32x^4y^2}$, $\sqrt{x^2+x^2}$.

6. Show that $\sqrt{4\times 9}$ is equal to $\sqrt{4}\times\sqrt{9}$, but that $\sqrt{4+9}$ is not equal to $\sqrt{4}+\sqrt{9}$, and that therefore $\sqrt{4}+\sqrt{9}$ is not equal to $\sqrt{13}$.

7. Write the following as powers of 10, and state the logarithm of each to the base 10:

   1000, 100, 1, 10,000, 1,000,000, 10 million.

8. Find the following:

   $\log_5 25$, $\log_2 32$, $\log_2 64$, $\log_4 64$, $\log_{\frac{1}{2}} \frac{1}{8}$, $\log_{-2} -8$.

9. Write down the product of the following in the form of powers:

   2^2 and 2^3, 2 and 2^4, 3^2 and 3^3, 5^4 and 25^3.

10. What is $\log_3 (3^2 \times 3^3)$?

Exercise IV (f)

Applications to Mensuration and Science

1. The perimeter of a figure is the distance round its boundaries.

   Draw the following figures, choose suitable symbols, a, b, c, etc., for the length of the sides, and write down in as simple a form as possible their perimeters.

   (i) A quadrilateral. (ii) A square. (iii) An oblong.
   (iv) A parallelogram. (v) A rhombus. (vi) A triangle.
   (vii) An isosceles triangle. (viii) An equilateral triangle.

2. Describe a circle, and draw a diameter and a radius. Call the diameter d and the radius r. The length of the circumference is found by multiplying the diameter by a number which is denoted by the Greek letter π (pi).

   Write down the circumference in terms of d and in terms of r.

3. Construct a semicircle and a quadrant of a circle, and write down the perimeter of each in terms of the radius.

4. Choose suitable symbols for the dimensions, and write down the area of each of the following:

   (i) A rectangle. (ii) A square.
   (iii) A parallelogram. (iv) A triangle.

5. Construct a trapezoid. Draw one of the diagonals, and the figure will be seen to consist of two triangles of the same altitude.

   If the lengths of the parallel sides are denoted by a and b, and their distance apart by h, find an expression for the area of the figure.

6. The area of a circle is π times the area of the square on the radius. Write this algebraically, and also the area of a quadrant of a circle.

7. Obtain expressions for the area of the shaded portions of figs. 7, 8 and 9.

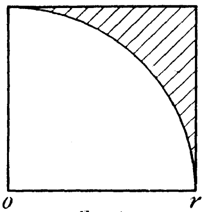


FIG. 7.

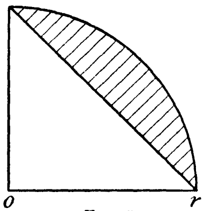


FIG. 8.

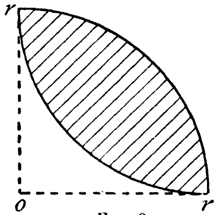
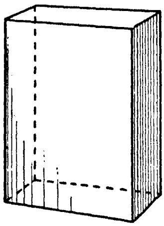


FIG. 9.

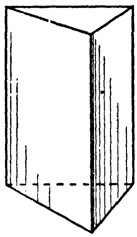
8. The volume of a prism is the product of the area of the base and the altitude of the prism.

Choose suitable symbols for the dimensions, and state algebraically the volume of the following:



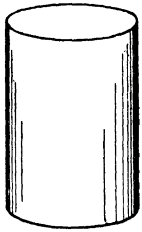
Rectangular Prism

FIG. 10.



Triangular Prism

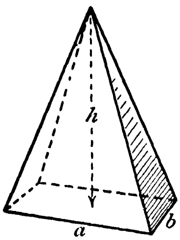
FIG. 11.



Cylinder

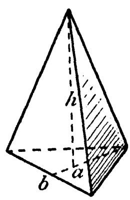
FIG. 12.

9. The volume of a pyramid is one-third that of the prism having the same base and altitude. Find the volume of the following:



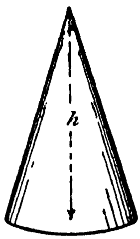
Rectangular Pyramid

FIG. 13.



Triangular Pyramid or Tetrahedron

FIG. 14.



Cone

FIG. 15.

10. In Ex. IV (B), No. 8, find also the gain in weight per gram of copper.

11. If x lead shot, all the same size, weigh y grams, what is the weight of one shot? How many such shot would there be in a quantity weighing p grams?

12. A wheel makes n turns in s seconds. How many does it make per second? How many will it make in t seconds?

13. The three angles of a triangle together measure 180°. If one is $x°$ and the remaining two are equal, what is the value of each?

14. A plank of wood measures l feet. It is cross-cut by a saw into three pieces of lengths a, b and c feet. Find the waste allowance in inches of each saw-cut.

15. If, in measuring a line of any length by a certain scale, the error is e mms., what will be the error in the result obtained for the number of cms. in an inch for which a line 10 inches long was measured?

§ 20. Representation by Graph.

The numerical value of, say, $4a$ depends, of course, upon the value of a. Thus, if a is 1, then $4a$ is 4; if a is 2, $4a$ is 8, and so on. The value of $4a$ changes then with the value of a. A change in the value of a from 1 to 2 makes a change in the value of $4a$ from 4 to 8.

There is a very convenient way of representing values which are subject to change, namely, by means of graphs.

You have probably met with it in your lessons in Arithmetic or Geography. Occasionally you meet with it in newspapers.

The following are examples:

i. Fig. 16 shows the amount of cotton imported by the United Kingdom during the years 1903 to 1912.

The years are marked on the horizontal line, and the length of each vertical line represents to scale the quantity of cotton imported during the year stated at the foot of the line. Thus, in 1906, the quantity was 20 million centals, i.e. 2000 million pounds.

The advantages of graphical representation are:

(1) At a glance, the amounts for each year can be compared.

(2) The diagram is more striking than a list of numbers.

(3) The lowest and the highest quantities are readily picked out.

(4) Some idea as to the importance of the changes (increases or decreases) can be quickly gained.

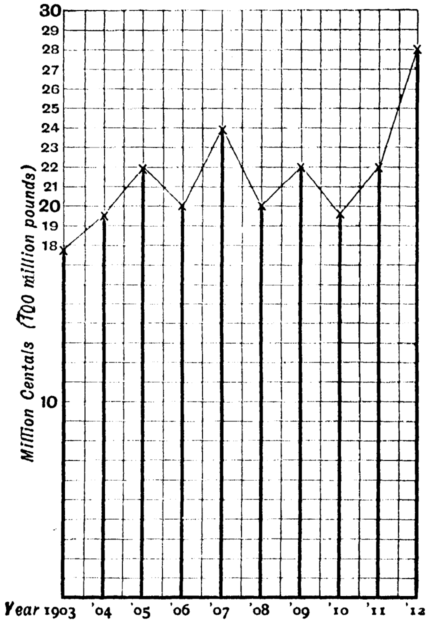


FIG. 16.

(5) An opinion can be readily formed as to whether there is, on the whole, an upward or a downward tendency.

(6) Exceptional cases stand out prominently.

(7) A close approximation to the average value is quickly made.

If the heads of the lines be joined by straight lines, rise and fall are strikingly distinguished.

ii. Height of the tide and the date.

Fig. 17 shows the height of successive tides and the date.

It is interesting to compare the graphs for several months, and also to see whether the changes agree with the phases of the moon. The wave form of this graph is apparent.

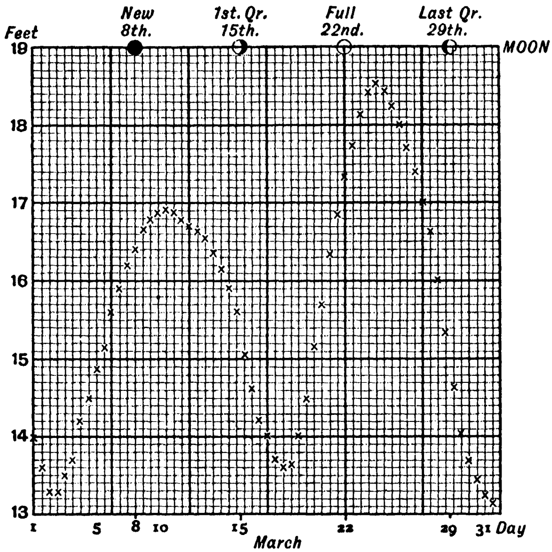


Fig. 17.

iii. Depth of a swimming bath and the length.

Fig. 18 shows the depths of a swimming bath at places along its length.

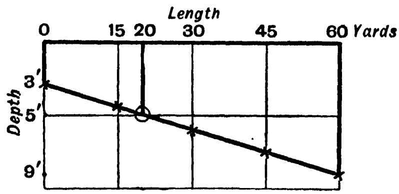


Fig. 18.

There is a matter concerning these graphs which is of the utmost importance: namely, under what conditions can a graph be used to determine values which are not definitely shown by marked points? Refer to the graph giving the depths of a swimming bath at various distances along its length.

The points showing the depth appear to follow a straight line. Draw the straight line, and drop the 'depth line' from the point denoting 20 yards, and read off what the depth appears to be at this place. Now this result is probably correct, but as to whether it is absolutely correct depends upon whether the change in depth, which we have found by measuring the depth at certain places, is uniform. The more measurements we make the better judgment we can form on this matter, and, of course, the matter could be decided definitely by examining the bath when empty. The correctness of the answer, found from the diagram, depends upon whether the change in depth found at certain places is *continuous* throughout the whole length of the bath.

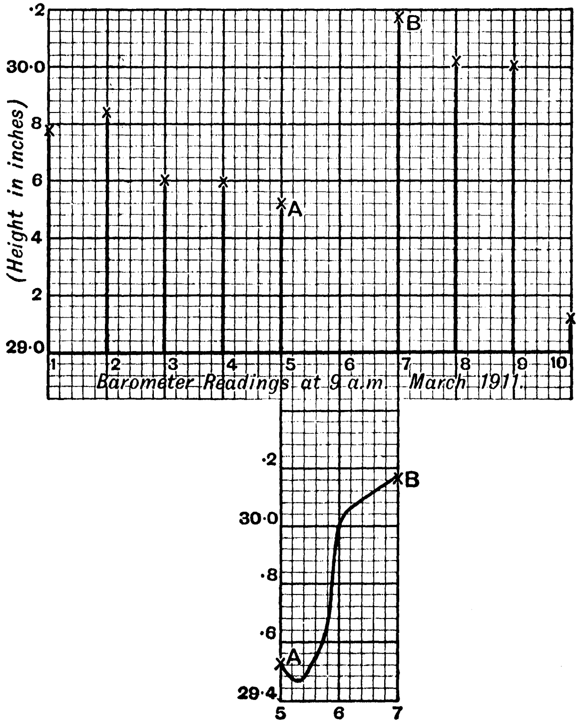


FIG. 19.

Turning now to the diagram of the readings of the barometer (fig. 19), it is observed that there appears to be no uniformity in

the sequence of the points. They do not lie on a smooth curve even, like those of diagrams 20 and 21.

You will notice that the reading for March 6th is missing. Can we, from the other readings, determine the missing reading? The behaviour of the barometer is so erratic that this is impossible. Joining A and B, the neighbouring points, by a straight line will not help us.

To see how difficult the problem is, it is only necessary to examine the curve of one of those instruments which, by means of a pen and clockwork, automatically draw a continuous graph for a day or a week. Such a curve is shown in the lower part of fig. 19. In it the missing reading will be found.

These various changes appear to follow no law. If there is a law, it is a very complex one.

Even where the points lie on a smooth curve or a straight line, as in fig. 20, which suggests that some law exists, we must

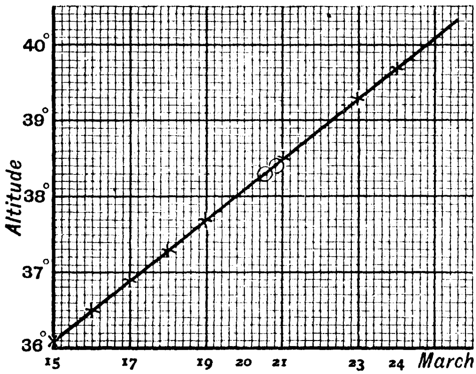


FIG. 20.

not conclude that the graph will give us values for intervals between the points for which it is constructed. E.g., referring to fig. 20, the graph of the sun's altitude at noon, if we draw the line midway between the two noons, March 20 and 21, that is, at a point which would represent midnight, the altitude obtained is 38°·3, which is, of course, ridiculous at the latitude of observation.

Again, at a point which would represent 6 a.m., the reading is 38°·4, which also is out of the question, since the sun at this time is just rising.

The changes for a part of the day between sunrise and sunset, March 21st, are shown in fig. 21.

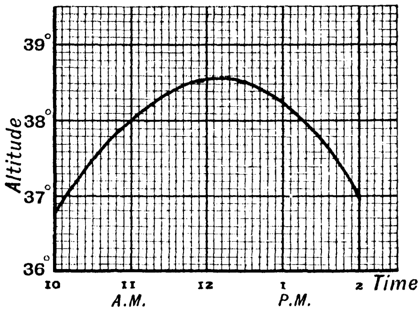


FIG. 21.

On the other hand, this figure can be used for values between the times shown, for the changes are continuous between the times of sunrise and sunset.

§ **21.** We can represent the various values of $4a$ in a similar manner.

Represent the values of a along the horizontal line, and the corresponding values of $4a$ along lines at right angles (fig. 22).

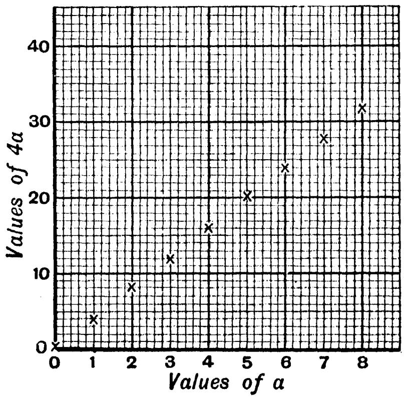


FIG. 22.

If squared paper is used, the lines are already drawn.

Arrange the values of a and of $4a$ in a table, thus:

$a=$	1	2	3	4	etc.
$4a=$	4	8	12	16	etc.

Mark the ends of the lines representing $4a$ by a cross (×).

Now, what line do these end points follow?

Draw the line.

See if this line enables you to find values of $4a$ not already calculated from values of a.

Try, for example, values of a beyond 10, or between 4 and 5.

Determine whether the line will hold for negative values of a.

State your conclusions clearly.

§ **22.** When two things are so associated or interdependent that one becomes definite when the other is made definite, the first is said to be a **function** of the second.

Thus the pressure of the atmosphere is a function of time. The weight of water is a function of the volume. The attendance at school may be regarded as a function of the weather. The value of gold is a function of the weight, and so on.

In many cases the real connexion between the things, or the law connecting them, is unknown. There are, however, some functions which can be expressed simply, and we shall deal with some of them later.

The value of $4a$ *is a function of* a.

When a definite value is given to a, the value of $4a$ is determined.

The graph gives correct values of $4a$ when a is positive, when a is zero and when a is negative. It is true when a is numerically great and when a is numerically small. Moreover, and this is important, we can find a value for a which will give any value we please for $4a$, i.e. it is possible for $4a$ to have any value whatever, and there will be a corresponding value for a also.

Exercise IV (G)

1. Examine carefully the graphs in figs. 16 to 22, and write down the conclusions you draw from each.
2. Make marks, say, a centimetre apart, on a wax candle. Screen the candle from draught, light it, and note the time at which the marks disappear.

Represent time on a horizontal line, and the length of the candle remaining at the noted time by lines at right angles to the time line.

After four or five observations, predict how long the candle will last.

Examine the graph, and draw your conclusions.

3. Fix a burette vertically in a stand. Fill the burette with coloured water, and turn the tap so that the liquid runs out slowly.

   Using a seconds' watch, take the time at which the surface of the liquid in the burette passes the 0, 5, 10, 15, etc., graduation marks.

   Construct a graph of your observations, examine it, and draw conclusions.

4. Take a steel spiral spring or a piece of rubber cord. Suspend it from a rigid bracket or nail, and to the lower end attach a light pan. In order to measure the length of the spring or cord, erect a ruler by its side. Increase gradually the weight applied, and note the length for each pull.

   Construct a graph of your results, and draw conclusions.

CHAPTER V

COMMON PROCESSES (*Continued*)

§1. Brackets.

We have seen on page 26 that when we wish terms, added or subtracted, to be considered together and not apart, they are bracketed.

Thus, $(3x - 2y)$ may be considered as one term, like the symbol a.

The bracket may have a coefficient, e.g.

$$5(3x - 2y) \quad \text{or} \quad -4(3x - 2y).$$

As a matter of fact, $(3x - 2y)$ has the coefficient 1, which it is not necessary to write.

Now the question is, suppose we wish to remove the bracket and separate the terms, what change will a coefficient make?

The meaning of $5(3x-2y)$ may be taken to be $(3x-2y)$ multiplied by 5, i.e. $3x \times 5 - 2y \times 5$ or $15x - 10y$; or, we may take it as meaning five such expressions added. Thus:

$$\begin{array}{r} 3x-2y \\ 3x-2y \\ 3x-2y \\ 3x-2y \\ 3x-2y \\ \hline 15x-10y \end{array}$$

The result is the same as before. The first is, however, the simpler operation.

Special cases.

i. $(3x-2y)$. The coefficient here is 1, and multiplying the terms by it does not alter them. Hence, $(3x-2y) = 3x - 2y$, without the bracket.

ii. $-(3x-2y)$. The coefficient here is -1, and the effect of multiplying by -1 is to change the signs. Thus: $-(3x-2y) = -3x + 2y$, without the bracket.

iii. $-4(3x+2y) = -12x - 8y$.

To show graphically that $-2(3x-2y) = -6x + 4y$.

Let x and y be represented by the lines shown.

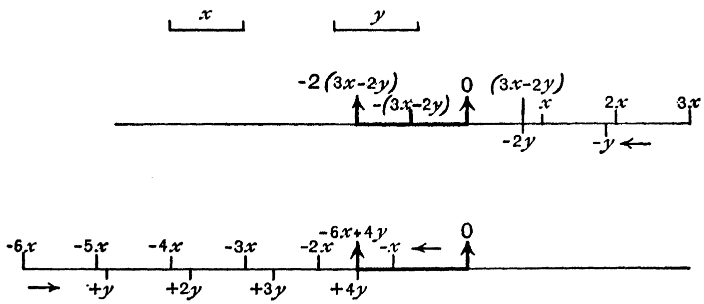


Fig. 1.

The upper long line shows $(3x-2y)$ on the right, and $-2(3x-2y)$ on the left of the zero.

The lower long line shows on the left of the zero $-6x+4y$.

It is seen that the two results $-2(3x-2y)$ and $-6x+4y$ are alike in magnitude and sign.

Note.—Had $(3x-2y)$ been to the left of the zero, $-2(3x-2y)$

would have been obtained by taking twice this distance on the right side of the zero.

§ 2. Insertion of Brackets.

In the reverse operation, to obtain the terms to be placed inside the bracket, divide the given terms by the number it is proposed to place outside. Thus:

EXAMPLE.—To bracket $-4x+6y$.

The terms can be divided by -2, giving $-2(2x-3y)$.

Incidentally, we have obtained the factors of $-4x+6y$, for -2 and $(2x-3y)$ when multiplied together give $-4x+6y$, and each is simpler than the product.

Note.—In an expression such as $\frac{2x-y}{2}$, the line separating the numerator from the denominator acts like a bracket. Not only $2x$, but $-y$ also, has to be divided by 2.

$\frac{2x-y}{2}$ may be written in the form $\frac{1}{2}(2x-y)$.

If the line or link be removed, the result is $\frac{2x}{2}-\frac{y}{2}$ or $x-\frac{y}{2}$.

The following case is very important.

Show that $(a-b)=-(b-a)$.

$$\begin{aligned}(a-b)&=a-b &&\text{(removing the bracket)}\\ &=-b+a &&\text{(rearranging the terms)}\\ &=-(b-a) &&\text{(bracketing, taking out the common factor } -1).\end{aligned}$$

EXERCISE V (A)

Show graphically that:

1. $2(2x-3y)=4x-6y$.
2. $-3(x-2y)=-3x+6y$.
3. $-(x-y)=-x+y$.
4. $-(x+y)=-x-y$.
5. Five bags each containing 30 sovereigns and a bill showing a debt of £8 are emptied out. State the total contents.

Verify your answer to each of the following exercises, by giving numerical values to the symbols.

Remove the brackets and simplify when possible

6. $5(2x-3y)$.
7. $-5(2x-3y)$.
8. $x(2x-3y)$.
9. $-x(2x-3y)$.
10. $-y(2x+3y)$.
11. $-y(2x-3y)$.

12. $a(a+b-c)+b(b+c-a)+c(a+c-b)$.

13. $3a-2(4b-5c)+3b-(4c-3a)-3c-(4a+3b)$.

14. $3(x-2y)-2(3x+4y)+x(3x-2y)$.

15. $(2a+3b)-2(3b-5a)$.

16. $2(3a+2b)-3(a-3b)-7(a+b)$.

17. $-(a-b)-(b-c)-(c-a)$.

Bracket the following; check your answers by removing the brackets.

18. $2x+2y$. 19. $-2x-2y$. 20. $-2x+6y$.

21. $3a-6b+12c$. 22. $3a-6b+5c+25d$.

23. $3a-6b-5c+25d$. 24. a^2-ab-c^2+cd.

25. $a^2+ab-ab+b^2$. 26. $x^3-3x^2y-xy^2$.

27. Show that $-(b-a)=(a-b)$.

Simplify:

28. $\dfrac{3x-6y}{3}$. 29. $\dfrac{15x^2+20y^2}{5}$. 30. $\dfrac{10y-4x}{-2}$.

§ 3. Complex Brackets.

It is sometimes found necessary to bracket terms which form part of an expression already bracketed.

The kinds of brackets used are:

(i) Square [ ]. (ii) Zig-zag { }.

(iii) Plain ( ). (iv) Vinculum, line or link ——.

An example will show the use of these brackets.

$$-2[6a+b+3\{8b-4(3-\overline{2a-b})\}].$$

You will notice that the innermost bracket is the vinculum, and the outermost the square bracket.

To simplify this expression, it is usual to commence with the innermost and to remove the brackets step by step.

Thus:

(i) $-2[6a+b+3\{8b-4(3-2a+b)\}]$. Vinculum removed.

(ii) $-2[6a+b+3\{8b-12+8a-4b\}]$. Plain bracket removed.

(iii) $-2[6a+b+24b-36+24a-12b]$. Zig-zag " "

(iv) $-12a-2b-48b+72-48a+24b$. Square " "

(v) $-60a-26b+72$. Terms collected.

§ **4.** As an exercise in the reverse process, the pupil should try to re-insert the brackets in the foregoing example, commencing from the last line but one.

Insert the square bracket first, and the others in successive steps

EXERCISE V (B)

Simplify:

1. $x - 2(x + 3y) - 3\{y + 2(2x - y)\}$.
2. $4b - [a - \{2a(x + y) - 3a(x - y)\}]$.
3. $6a - [2a - \{3a - (5a + 3)\}]$.
4. $12x - 2[x - 3\{x - 4(x - \overline{2 - x})\} + x]$.
5. $2[10 - \{8 + (3 - 6) - 2\} + 5] + 3$.
6. $2x - [3y - \{x + 2(3y + 1)\} + 2\{2y - 3(2x - 1)\} + 2x]$.

Insert brackets where possible in the following:

7. $6a^2 - 2a^2b + 2ab^2$.
8. $a^2x^2 + axy - a^2y^2$.
9. $ab + ac - abc$.
10. $ax^2 + bx - cy^2 + dy$.
11. $a(x + y) + ac$.
12. $(a + b)(x + a) - (a + b)(x + b)$.
13. $(p + q)(x + y) + (p + q)c$.
14. $(p + q)(x + y) + (p + q)(y + z)$.
15. $(p + q)(x + y) - (p + q)(x - y)$.
16. $a^2 + ab + ab + b^2$.
17. The long and short sides of a rectangle or oblong are respectively a and b units in length. Represent the perimeter in as simple a form as possible.

    Find the value of your result when $a = 3{\cdot}6$ inches and $b = 2{\cdot}5$ inches.
18. The length, breadth and height of a rectangular room are respectively a, b and h. Find an expression for the total area of the walls, floor and ceiling.
19. $$\frac{\pi h a^2}{3} + \frac{\pi h a b}{6} + \frac{\pi h b^2}{3}.$$

    Find the value when $h = 6{\cdot}32$, $a = 4$ and $b = 3$.
20. Given that $\sqrt{3} = 1{\cdot}732\ldots$, evaluate in the shortest way
$$8\sqrt{3} + 2{\cdot}56\sqrt{3} - 7\sqrt{3} - 1{\cdot}36\sqrt{3}.$$

§ 5. Factors of Simple Terms.

The factors of a term or expression are simpler terms or expressions which when multiplied together produce the term or expression.

The simplest factors are those which cannot be split up into simpler factors. *In Arithmetic, such factors are called Prime Factors.*

The factors of a^2bx are a, a, b, x, because $a \times a \times b \times x$ equals a^2bx, and each is simpler than a^2bx.

The prime factors of 30 are $2 \times 3 \times 5$, because $2 \times 3 \times 5 = 30$, and none of the numbers 2, 3 and 5 can be split into simpler factors.

The factors of $-36a^2b^3$ are -2, 2, 3, 3, a, a, b, b, b, because

$$-2 \times 2 \times 3 \times 3 \times a \times a \times b \times b \times b = -36a^2b^3.$$

It is unnecessary to write the same factors at such length, the following shorter form is quite good: $-(2)^2$, $(3)^2$, a^2, b^3.

§ 6. Highest Common Factor and Lowest Common Multiple.

It is an easy matter to determine the Highest Common Factor (H.C.F.) in a set of algebraic terms.

Thus, in the terms a^3b^2c, $a^2b^3c^2$, $-a^4bc^3d^2$, it is readily seen that the highest power of a which is a factor of all three terms is a^2, that the highest power of b which is a factor of all the terms is b, and that the highest power of c, common to all three terms, is c; also that d is not a common factor, since d to the first or any higher power appears in neither the first nor the second term.

The H.C.F. is therefore a^2bc.

It will be noticed that the H.C.F. contains the lowest power of each symbol contained in all the terms. E.g. a is contained in all the terms, the lowest power being a^2, which appears in the second term.

The result can be checked by dividing each term by the H.C.F. The quotients should have no common factor other than unity.

$$\frac{a^3b^2c}{a^2bc} = ab,$$

$$\frac{a^2b^3c^2}{a^2bc} = b^2c,$$

$$\frac{-a^4bc^3d^2}{a^2bc} = -a^2c^2d^2.$$

Neither a, b, c nor d is contained in all three quotients.

If there are coefficients, of course the H.C.F. of these also must be found.

Thus the H.C.F. of $16a^3b^2c$, $24a^2b^3c^2$, $-20a^4bc^3d^2$ is $4a^2bc$.

§ **7.** The determination of the Lowest Common Multiple (L.C.M.) of a number of algebraic terms is equally simple.

EXAMPLE.—Find the L.C.M. of

$$a^3b^2c, \quad a^2b^3c^2 \quad \text{and} \quad -a^4bc^3d^2.$$

The answer must contain every symbol in the terms; but the result, to be the *lowest* multiple, must not contain any power of a symbol higher than the highest appearing in the terms. Thus, a^4 in the third term is the highest power of a in all three terms. Hence the L.C.M. must of necessity contain a^4, but no higher power; for the same reason the L.C.M. must contain b^3, c^3 and d^2. The L.C.M. is therefore $a^4b^3c^3d^2$.

Any L.C.M. can be checked by dividing it by each of the terms. The quotients should have no common factor other than unity.

$$\frac{a^4b^3c^3d^2}{a^3b^2c} = abc^2d^2,$$

$$\frac{a^4b^3c^3d^2}{a^2b^3c^2} = a^2cd^2,$$

$$\frac{a^4b^3c^3d^2}{-a^4bc^3d^2} = b^2.$$

Neither a, b, c nor d appears in all three quotients.

If there are coefficients, the L.C.M. of the numbers also must be found. Thus, the L.C.M. of

$$16a^3b^2c, \quad 24a^2b^3c^2, \quad -20a^4bc^3d^2 \quad \text{is} \quad 240a^4b^3c^3d^2.$$

Note.—In H.C.F. and L.C.M. the result may be either positive or negative. It is usual, however, to give the positive value only.

§ 8. Easy Fractions.

The four processes, addition, subtraction, multiplication and division, are carried out in Algebra in the same way as in Arithmetic.

Addition and Subtraction.

EXAMPLE i.—Find

$$\frac{3}{2bc} - \frac{1}{ac} + \frac{2}{3ab}$$

$$= \frac{9a - 6b + 4c}{\text{L.C.M., } 6abc}.$$

Find the L.C.M. of the denominators.

Reduce all the fractions to this common denominator, e.g. in the case of $\frac{3}{2bc}$, divide $2bc$ into $6abc$ and multiply the numerator 3 by the result, viz. $3a$.

EXAMPLE ii.

$$\frac{a+2b}{3c}-\frac{b-3c}{2a}-\frac{4ab-3bc}{6ac}$$

Notice that the numerators are bracketed by a vinculum.

$$=\frac{2a(a+2b)-3c(b-3c)-(4ab-3bc)}{\text{L.C.M., }6ac}$$

Note the plain brackets.

$$=\frac{2a^2+4ab-3bc+9c^2-4ab+3bc}{6ac}$$

Notice the change of signs in some cases.

$$=\frac{2a^2+9c^2}{6ac}.$$

Multiplication and Division.

EXAMPLE iii. $\dfrac{a^2b}{cd^2}\times\dfrac{a}{c}=\dfrac{a^2b\times a}{cd^2\times c}=\dfrac{a^3b}{c^2d^2}.$

EXAMPLE iv. $\dfrac{2a^2b}{3cd^2}\div\dfrac{4a}{9c}=\dfrac{\overset{ab}{\cancel{2a^2b}}}{\underset{d^2}{\cancel{3cd^2}}}\times\dfrac{\overset{3}{\cancel{9c}}}{\underset{2}{\cancel{4a}}}=\dfrac{3ab}{2d^2}.$

Observe that cancelling, that is, the dividing of a numerator and a denominator by a common factor, is carried out wherever possible.

EXERCISE V (c)

Find the H.C.F. and L.C.M. of :

1. x^2y^2 and x^3y.
2. $a^2b^3c^4$ and ab^2c^3.
3. a^4 and a^3b^3.
4. $10x^4y^3$ and $15x^3y^4$.
5. $3ab^2c^3$ and $6a^2bc^2$.
6. $a^4b^5c^3$, $a^3b^4c^5$ and $a^5b^3c^4$.
7. $2bc$, ac, $3ab$.
8. $3c$, $2a$, $6ac$.
9. 25, $5ab$ and $10bc$.

Simplify :

10. $\dfrac{ax+bx}{x}$.
11. $\dfrac{10a+6ad}{2a}$.
12. $\dfrac{3x+3y}{2x+2y}$.
13. $\dfrac{a}{x}+\dfrac{2}{x}$.
14. $\dfrac{a}{x}+\dfrac{b}{2x}$.
15. $\dfrac{x}{a}+\dfrac{a}{x}$.
16. $\dfrac{a-b}{2b-2a}$.
17. $\dfrac{x+y}{a-b}-\dfrac{2x-y}{2a-2b}$.
18. $\dfrac{a}{b}\times\dfrac{b}{c}$.
19. $\dfrac{ab}{cd}\div-\dfrac{a}{d}$.
20. $\dfrac{3x}{4y}+\dfrac{2y}{3x}-\dfrac{5x}{12y}$.

21. $\dfrac{x}{2y}+\dfrac{2y}{3x}$.

22. $\dfrac{5a+1}{6}-\dfrac{a+1}{2}-\dfrac{3a+1}{4}$.

23. $\dfrac{2y-3}{3}-\dfrac{3y+5}{5}+\dfrac{5y+3}{6}-\dfrac{7x+5}{10}$.

24. $\dfrac{ax+ay}{3bx}\div\dfrac{3x+3y}{6ax}$.

25. $\dfrac{\dfrac{3ax-6bx}{7a^2b^2}\times\dfrac{3ab}{6xy}}{\dfrac{ay-2by}{14bx}}$.

26. $(5\sqrt{3}-2\sqrt{3})\times(4\sqrt{3}+\sqrt{3})$.

27. $\dfrac{2ax+2}{2ax}\times\dfrac{3bx}{3bx+3}$.

28. $\dfrac{ab}{ab+b^2}\div\dfrac{b}{a+b}$.

29. Divide $ab(ab+b^2)$ by b.

30. Multiply $3b(a+1)$ by $2a$.

CHAPTER VI

SIMPLE EQUATIONS

§ **1.** EXAMPLE i.—Consider the following little problem: *A bag of sugar and 2 pound weights in one pan of a balance, balance 6 pound weights in the other. What is the weight of the bag of sugar?**

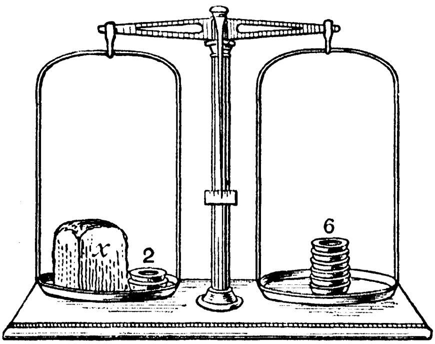


FIG. 1.

Now, in weighing, we usually put the thing we are weighing, alone in one pan.

* Packets of pen-nibs counterbalanced by loose pen-nibs will be found useful for practical exercises. The results can be verified by opening the packets.

What would be the effect of taking the 2 pound weights off the left-hand pan?

What would you do with the other side to restore the balance?

You would, of course, take 2 pound weights off.

The bag of sugar would then be seen to weigh 4 pounds.

This can be put down very conveniently in Algebra.

Write x for the number of pounds that the bag of sugar weighs.

It is now our business to find x.

We are told that $x+2$ equals 6; written

$$x+2=6.$$

This is called an equation, and from it we have to find the value of x.

To get x left alone on one side, subtract 2 from each side; then

$$x=6-2;$$

$$\text{i.e. } x=4.$$

You will observe that if we consider the number 2 to have been carried to the other side of the equation, its sign has been changed from + to −. If we imagine it put back, the sign must be changed back from − to +. This is an important rule, viz.: *When an added or subtracted number is taken from one side of an equation to the other, its sign must be changed.*

§ **2.** Example ii.—*If $x-2=6$, find x.*

The minus 2 can be got rid of by adding plus 2 to each side, thus:

$$x-2+2=6+2,$$

$$\text{i.e. } x=6+2,$$

$$\text{i.e. } x=8.$$

Putting 8 instead of x in the original equation the result is seen to be correct.

Here again we might have taken the -2 to the other side and changed it to $+2$.

§ **3.** Example iii.—Being told that three times a chosen number was 6, we should at once conclude that the chosen number was 2.

To state this in algebraic form, let the chosen number be represented by the symbol x; then

$$3x=6.$$

The value of x is obtained at once by taking a third of 6,

$$\text{i.e. } x = \tfrac{6}{3}$$
$$= 2.$$

Actually, both sides of the equation have been divided by 3, for $\frac{3x}{3} = x$. It is common experience that thirds of equal things are equal.

The result can be verified by substituting the value obtained in the original equation, thus:

When x is 2, $3x$ does equal 6.

Similarly, if $$12x = -60,$$
$$x = \frac{-60}{12},$$
$$\text{i.e. } x = -5.$$

Check, $$12x = 12 \times -5 = -60.$$

Also, if $$-12x = -66,$$
$$x = \frac{-66}{-12},$$
$$\text{i.e. } x = 5\tfrac{1}{2}.$$

Check, $$12x = -12 \times 5\tfrac{1}{2} = -66.$$

Observe that the coefficient of x becomes a divisor on the other side of the equation.

§ **4.** Example iv.—An equation may have the symbol for the unknown number on both sides.

In such cases arrange the equation so that all such symbols are on the same side. Remember the rule given on page 56.

$$5x - 13 = 3x + 5,$$
$$5x - 3x = 5 + 13,$$
$$2x = 18,$$
$$x = 9.$$

Check,

Left side: $5x - 13 = 5 \times 9 - 13 = 45 - 13 = 32.$

Right side: $3x + 5 = 3 \times 9 + 5 = 27 + 5 = 32.$

The sides are equal when x is 9.

Exercise VI (a)

Find the value of the symbol of the following equations, and explain each step.

1. $6=x+2$.	2. $x-3=6$.	3. $x-3=-6$.
4. $x+5=5$.	5. $x-5=-5$.	6. $x+5=-5$.
7. $4+x=9$.	8. $4-x=9$.	9. $9=4-x$.
10. $6x=-18$.	11. $-6x=18$.	12. $-6x=-18$.
13. $3x=10$.	14. $-3x=10$.	15. $3x=-10$.
16. $-3x=-10$.	17. $4x=\frac{1}{2}$.	18. $-4x=\frac{1}{2}$.
19. $-4x=-\frac{1}{2}$.	20. $4x=-\frac{1}{2}$.	21. $6x+5=-13$.
22. $6x+18=0$.	23. $-5x-3=12$.	24. $2(x-3)=4$.
25. $x-3=4-(x-3)$.	26. $-7(x-3)=14$.	27. $3x-11=4-2x$.

§ **5.** Example v.—*If* $\frac{1}{2}x=6$, *find* x.

It is common knowledge that to whatever number half x is equal, x is equal to twice that number.

In this case $x=12$.

Actually, both sides of the given equation have been multiplied by 2.

Thus:
$$\tfrac{1}{2}x\times 2=6\times 2,$$
$$\text{i.e. } x=12.$$

Generally, *the equality is not destroyed by multiplying or dividing both sides of an equation by the same number.*

The operation performed above is really the same as that of Example iii.

Thus:
$$\tfrac{1}{2}x=6;$$
$$\therefore\ x=\frac{6}{\frac{1}{2}}=6\times\tfrac{2}{1}=12.$$

Similarly, if
$$-\tfrac{3}{5}x=18,$$
$$x=18\times-\tfrac{5}{3},$$
$$\text{i.e. } x=-30.$$

Check,
$$-\tfrac{3}{5}x=-\tfrac{3}{5}\times-30=18.$$

Notice that the 5, which is a divisor on the left, becomes a multiplier on the right, and that the 3, which is a multiplier on the left, becomes a divisor on the right.

§ 6. EXAMPLE vi.—The symbol need not necessarily be on the left side, thus:

If $$18 = 3x,$$

then $$\frac{18}{3} = x,$$

i.e. $$6 = x.$$

§ 7. EXAMPLE vii.

$$\frac{15}{x} = -3,$$

Multiply each side by x, or make x a multiplier on the right.

$$15 = -3x,$$

$$\frac{15}{-3} = x,$$

i.e. $$-5 = x.$$

Another method is to invert both sides of the equation, then:

$$\frac{x}{15} = -\frac{1}{3},$$

$$x = -\frac{15}{3},$$

i.e. $$x = -5.$$

Check, $$\frac{15}{x} = \frac{15}{-5} = -3.$$

§ 8. EXAMPLE viii.—*Solve the equation:*

$$3(x-2)+5 = \frac{3x-5}{2} + 6.$$

Multiply both sides by 2, to get rid of the denominator 2.

$$6(x-2)+10 = (3x-5)+12,$$

Remove the brackets.

$$6x - 12 + 10 = 3x - 5 + 12,$$

Arrange the sides.

$$6x - 3x = 12 - 10 - 5 + 12,$$

$$3x = 9,$$

$$x = 3.$$

Check,

Left side: $3(x-2)+5 = 3(3-2)+5 = 3+5 = 8.$

Right side: $\frac{3x-5}{2} + 6 = \frac{3 \times 3 - 5}{2} + 6 = 2 + 6 = 8.$

Exercise VI (b)

Find the value of the symbol in the following equations. Check each value.

1. $\frac{1}{3}x = 2$. 2. $\frac{2}{3}x = 2$. 3. $-\frac{2}{3}x = 2$. 4. $-\frac{2}{3}x = -2$.

5. $\frac{1}{3}x = \frac{1}{6}$. 6. $1\frac{1}{2}x = 6$. 7. $-1\frac{1}{2}x = 6$. 8. $-3{\cdot}5x = 14$.

9. $\frac{6}{x} = \frac{1}{2}$. 10. $\frac{6}{x} = 2$. 11. $-\frac{6}{x} = 2$. 12. $\frac{2}{x} = 6$.

13. $\frac{5}{x} + 3 = \frac{2}{x}$. 14. $\frac{6}{2x} = 2$. 15. $\frac{5}{3x} = \frac{5}{3}$.

16. $\frac{5}{3x} = \frac{3}{5}$. 17. $\frac{5x}{3} = \frac{3}{5}$. 18. $-\frac{7}{x} + 1 = \frac{2}{x} + 4$.

19. $1 + \frac{1}{2}x = 6$. 20. $\frac{1}{2}\left(4 - \frac{2}{x}\right) = 3$.

21. $2x - 78 = 23 - 3x$. 22. $3(x - 14) = 7(x - 18)$.

23. $7(5 - x) = 8(x - 5)$. 24. $\frac{2y + 12}{6} = \frac{7y + 108}{24}$.

25. $\frac{x + 36}{3} = \frac{2x}{5} + 3$. 26. $x - \frac{19 - 2x}{4} = \frac{2x + 11}{2}$.

27. $\frac{1}{3}\left(\frac{x}{8} - 2\right) - \frac{2}{3}\left(\frac{x}{6} - 4\right) = \frac{2}{9}\left(\frac{x}{4} - 6\right)$. 28. $\frac{3x}{2} - \frac{2x - 3}{3} = \frac{x - 3}{4}$.

29. $\frac{x}{3} - \frac{x}{4} + \frac{1}{6} = \frac{x}{8} + \frac{1}{12}$. 30. $\frac{1}{3x - 5} = \frac{2}{x + 5}$.

31. The following is a useful application of simple equations:

1000 farthings = 960 farthings + 40 farthings,

i.e. 1000 farthings = £1 + 10d.

Hence, 1 farthing = £·001 + ·01d. (dividing by 1000).

The cost of 100 articles at, say, $4\frac{1}{4}$d. each is readily calculated as follows:

$4\frac{1}{4}$d. = 17 farthings = £·017 + ·17d.

Hence, the required cost = £1·7 + 17d. = £1. 15s. 5d.

Similarly, find the cost of 100 articles at 2s. $4\frac{1}{2}$d. each. (Remember that a florin is £·1.)

32. Knowing that £·1 = a florin, £·05 = 1 shilling and £·025 = 6d., express in decimal form:

(i) £3. 3s. $6\frac{1}{4}$d.,
(ii) £3. 7s. $6\frac{1}{2}$d.,
(iii) £3. 17s. $6\frac{1}{2}$d.,
(iv) £3. 7s. $8\frac{1}{4}$d.,

and quickly find the cost of 200 articles at these prices each.

33. The time (T) at any longitude (L) may be calculated from the time at Greenwich by means of the simple equation:

$T = G \pm \frac{L}{15}$, where T is the time required (in hours),
G is the time at Greenwich (in hours),
L is the longitude (in degrees).
When East, take the + sign.
When West, take the − sign.

Find the time at

(i) 45° E. when it is 8 a.m. at Greenwich.
(ii) 45° W. when it is 8 a.m. at Greenwich.
(iii) 75° W. when it is 12 (noon) at Greenwich.
(iv) 75° E. when it is 12 (noon) at Greenwich.
(v) 80° E. when it is 6 p.m. at Greenwich.

CHAPTER VII

MULTIPLICATION AND DIVISION OF EXPRESSIONS, SQUARE AND SQUARE ROOT

§ 1. Multiplication.

It should be clearly realised that $(a+b)(c+d)$ means that each term of $(c+d)$ has to be multiplied by a and also by b and the products added.

EXAMPLE i.—$(a+b)(c+d) = a(c+d) + b(c+d)$
$= ac + ad + bc + bd.$

The operation might have been set out as follows:

$$\begin{array}{rl} & c+d \\ & \underline{a+b} \\ \text{Multiply by } a, & ac+ad \\ \text{Multiply by } b, & \underline{\qquad\quad +bc+bd} \\ \text{Add,} & \underline{ac+ad+bc+bd} \end{array}$$

For complex expressions the latter arrangement is preferable, but for simple expressions we shall adopt the former.

Fig. 1 illustrates the result.

EXAMPLE ii.—$(a-b)(c-d) = a(c-d) - b(c-d)$
$= ac - ad - bc + bd.$

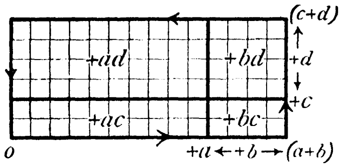


FIG. 1.

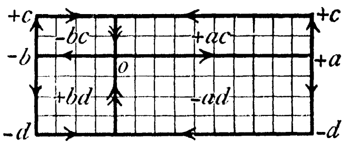


FIG. 2.

EXAMPLE iii.—$(x^2-3xy+4y^2)(x^2-2xy-2y^2)$.

$$\begin{array}{rl} & x^2-3xy\ +4y^2 \\ & \underline{x^2-2xy\ -2y^2} \\ \text{Multiply by } x^2, & x^4-3x^3y+4x^2y^2 \\ \text{Multiply by } -2xy, & \qquad -2x^3y+6x^2y^2-8xy^3 \\ \text{Multiply by } -2y^2, & \qquad\qquad\qquad -2x^2y^2+6xy^3-8y^4 \\ \text{Add,} & \underline{x^4-5x^3y+8x^2y^2-2xy^3-8y^4} \end{array}$$

Notice that like terms are placed in the same column.

EXERCISE VII (A)

Check your results by giving numerical values to the symbols:

1. $(a+b)(c-d)$.
2. $(a-b)(c+d)$.
3. $(2a+3b)(3c+2d)$.
4. $(2a+3b)(3c-2d)$.
5. $(2a-3b)(3c-2d)$.
6. $(3m^2-5n^2)(2p^3-3q^3)$.
7. $(x-2)(x-3)$.
8. $(x+2)(x+3)$.
9. $(x+2)(x-3)$.
10. $(x-2)(x+3)$.

11. $(2x+y)(2x-3y)$.
12. $2(3x-2)(2x-3)$.
13. $a(x-a)$.
14. $x(x-a)(x+a)$.
15. $(a+b)(c+d+e)$.
16. $(a+b)(a-b+c)$.
17. $(a+3)(b-4)$.
18. $(2xy-5)(5xy+3)$.
19. $(x^2+3x-1)(x-2)$.
20. $(2a^2-3a+2)(a^2-2a+1)$.
21. $(x^2+3xy-4y^2)(x^2-2xy-3y^2)$.
22. $\left(x^2+1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)$.
23. $(a^2+b^2+c^2+bc-ac+ab)(a-b+c)$.
Test your answer by putting b and c each equal to a.
24. Multiply $2+3x-4x^2+2x^3$ by $2-x+x^2-3x^3$.

The following products are very important, and should therefore be remembered. They are illustrated graphically in figs. 3, 4 and 5.

$$\begin{aligned}(1)\quad (x+y)(x+y) &= x(x+y)+y(x+y)\\ &= x^2+xy+xy+y^2\\ &= x^2+2xy+y^2.\end{aligned}$$

This is, of course, the square of $(x+y)$, and therefore we can write:

$$(x+y)^2 = x^2+2xy+y^2.$$

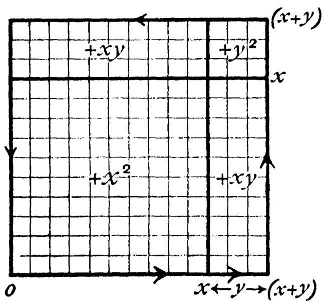


FIG. 3.

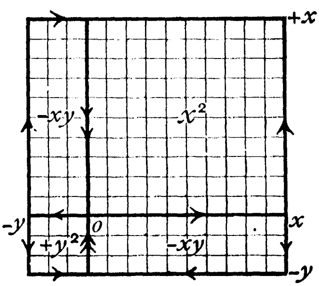


FIG. 4.

$$\begin{aligned}(2)\quad (x-y)(x-y) &= x(x-y)-y(x-y)\\ &= x^2-xy-xy+y^2\\ &= x^2-2xy+y^2,\end{aligned}$$

i.e. $(x-y)^2 = x^2-2xy+y^2$.

Notice that the expression obtained by squaring an expression of two terms (called a *binomial*) contains:

The square of the first term.
Twice the product of the two terms.
The square of the second term.

$$(3) \quad (x+y)(x-y) = x(x-y) + y(x-y)$$
$$= x^2 - xy + xy - y^2$$
$$= x^2 - y^2.$$

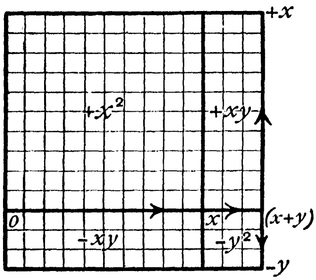


FIG. 5.

This result may be stated as follows:

The product of the sum of and the difference between two terms is equal to the difference between the squares of the terms.

Identity.

An equation such as:

$$(x+y)^2 = x^2 + 2xy + y^2,$$

which consists of the same statement in two different forms, is called an *Identity*. It is true for all values of the symbols.

Contrast this with an equation like $x + 3 = 5$, which is true for particular values of x only—in this case 2.

EXERCISE VII (B)

Find:

1. $(a+b)^2$, $(a-b)^2$, $(a+b)(a-b)$.
2. $(2a+b)^2$, $(2a-b)^2$, $(2a+b)(2a-b)$.
3. $(a+2b)^2$, $(a-2b)^2$, $(a+2b)(a-2b)$.

4. $(2x+5)^2,\ (2x-5)^2,\ (2x+5)(2x-5).$

5. $\left(\frac{x}{2}+1\right)^2,\ \left(\frac{x}{2}-1\right)^2,\ \left(\frac{x}{2}+1\right)\left(\frac{x}{2}-1\right).$

6. $(2x+3y)^2,\ (2x-3y)^2,\ (2x+3y)(2x-3y).$

7. $(10+7)^2,\ (50-3)^2,\ (50+5)(50-5),\ (50+\cdot 03)^2.$

8. $\{(a+b)+c\}^2,\ \{(a+b)-c\}^2,\ \{(a+b)+c\}\{(a+b)-c\}.$

9. $\{(2a-b)+2c\}^2,\ \{(2a-b)-2c\}^2,$
$$\{(2a-b)+2c\}\{2a-b)-2c\}.$$

10. $\{6(2a+b)\}^2,\ \{(a+b)(a-b)\}^2.$

11. Choose several different values for x and y, and verify that in every case
$$(x+y)^2 = x^2 + 2xy + y^2.$$

§2. Division.

The arrangement for division of algebraic expressions is like that used in Arithmetic.

EXAMPLE.—*Divide* $6x^2 + 11xy - 10y^2$ *by* $2x + 5y$.

(i) For the first term of the answer, divide $6x^2$ by $2x$.

(ii) Multiply the whole of the divisor by $3x$, place the result under the dividend, and subtract.

(iii) For the next term, divide $-4xy$ by $2x$.

Continue as in (ii).

Repeat these operations until either there is no remainder or a remainder which is simpler than the divisor.

$$\begin{array}{r|l} & 3x \quad - 2y \quad (\textit{Quotient}) \\ 2x+5y & 6x^2 + 11xy - 10y^2 \quad (\textit{Dividend}) \\ & 6x^2 + 15xy \\ \hline & \quad - 4xy - 10y^2 \\ & \quad - 4xy - 10y^2 \\ \hline & \quad \cdot \qquad \cdot \end{array}$$

Notice that the dividend and divisor have their symbols in the same order.

The result may be checked either by multiplying the quotient and divisor, and comparing the product with the dividend, or by giving numerical values to the symbols, finding the values of the divisor, dividend and quotient, and checking by Arithmetic.

Exercise VII (c)

1. Verify by two methods the correctness of the worked example.

Divide :

2. $a^2+2ab+b^2$ by $a+b$.
3. $a^2-2ab+b^2$ by $a-b$
4. a^2-b^2 by $a+b$.
5. a^2-b^2 by $a-b$.
6. a^2+b^2 by $a+b$.
7. a^2+b^2 by $a-b$.
8. x^2-5x+6 by $x-3$.
9. $4a^3-8a^2+8a$ by $4a$.
10. $-15x^3y^3-5x^2y^2+20xy$ by $-5xy$.
11. $2x^3-x^2+3x-9$ by $2x-3$.
12. a^4-16b^4 by $a+2b$.
13. $12+a-5a^2+a^3$ by $4-a$.
14. $6x^4-x^3y-x^2y^2+11xy^3-15y^4$ by $3x^2-2xy+5y^2$.
15. x^4+64 by x^2-4x+8.
16. $1-a-3a^2-a^5$ by $1-3a+2a^2-a^3$.
17. $6x^4-x^3-9x^2+9x-5$ by $3x^2-2x+1$.

    What is the remainder? Now find for what value of x this remainder will equal 0.
18. Find c such that x^2+5x+c is exactly divisible by $x+2$.

§ 3. The Square of a Binomial.

Given the first two terms of the square of a binomial to find the third term.

Example. x^2-6xy.

(i) Since x^2 is the square of the first term of the binomial, the first term is x.

(ii) Since $-6xy$ is twice the product of the two terms, the product is $\frac{-6xy}{2}$, i.e. $-3xy$, and since one term is x, the other is $\frac{-3xy}{x}$, i.e. $-3y$.

(iii) The binomial is therefore $(x-3y)$ and the complete square $x^2-6xy+9y^2$.

More briefly : To find the third term, divide the second term by twice the square root of the first term and square the result. Thus :

$$\frac{-6xy}{2x}=-3y,$$

$$(-3y)^2=9y^2.$$

The process is illustrated in fig. 6.

The large rectangle represents $x^2 + ax$. Halve the rectangle which represents ax, by a straight line which bisects a. Place the upper half of the rectangle in the position indicated by the

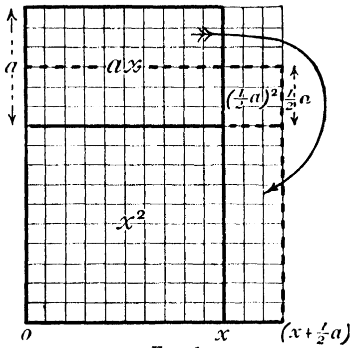


FIG. 6.

arrow. It is then seen that a square of side $\frac{1}{2}a$, and therefore of area $\frac{1}{4}a^2$, is required to complete the square, the side of which is $(x + \frac{1}{2}a)$.

EXERCISE VII (D)

Find the terms necessary to make each of the following expressions the square of a binomial. State the binomial in each case.

1. (i) $a^2 + 2ab$.	(ii) $a^2 - 2ab$.	(iii) $a^2 \ldots + b^2$.
2. (i) $x^2 + 4xy$.	(ii) $x^2 - 4xy$.	(iii) $x^2 \ldots + 4y^2$.
3. (i) $2ab + b^2$.	(ii) $-2ab + b^2$.	(iii) $-2b + b^2$.
4. (i) $4y^2 - 4xy$.	(ii) $4x^2 + 4xy$.	(iii) $4x^2 + 4x$.
5. (i) $9x^2 - 12xy$.	(ii) $9x^2 + 12xy$.	(iii) $9x^2 - 12x$.
6. (i) $x^2 + x$.	(ii) $x^2 - x$.	(iii) $x^2 - \cdot 8x$.

7. $49x^4 - 70x^2y^2$. 8. $a^2x^2 - 6ax$.

9. $a^2x^2 + 6axby$. 10. $9x^4 - 6x^2y$.

11. $16a^6 - 8a^3b$. 12. $\dfrac{9}{x^2} - \dfrac{12}{xy}$.

§ 4. Square Root.

We have seen that:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

It follows that the square root of $a^2 + 2ab + b^2$ is $a + b$,

$$\text{i.e. } \sqrt{a^2 + 2ab + b^2} = a + b.$$

Examining $a^2+2ab+b^2$, we see that:

(i) The first term (a) of the square root is the square root of the first term (a^2) of the expression.

(ii) The second term (b) of the square root is contained in the remaining part of the expression, $2ab+b^2$, which may be written $b(2a+b)$.

The process of finding square root is arranged as follows:

(i) The first term a, is the square root of a^2.

(ii) Subtract a^2 from the expression.

(iii) Form a new divisor by doubling what is in the answer (a), and adding the result (b) of dividing this double $(2a)$, into the first term $(2ab)$, of the line $2ab+b^2$.

(iv) Place the quotient (b), in the answer, multiply the new divisor by it, and complete the step as in division.

$$\begin{array}{rl}
 & a+b \quad \textit{Answer.} \\
a\;) & a^2+2ab+b^2 \\
 & a^2 \\
2a+b\;) & \quad +2ab+b^2 \\
 & \quad \underline{+2ab+b^2} \\
 & \quad \cdot \qquad \cdot
\end{array}$$

General Example.—*Find* $\sqrt{9x^4-12x^3-2x^2+4x+1}$.

First divisor,

$\sqrt{9x^4}=3x^2$.

Second divisor,

$2\times 3x^2+\dfrac{-12x^3}{6x^2}$.

Third divisor,

$2(3x^2-2x)+\dfrac{-6x^2}{6x^2}$.

$$\begin{array}{rl}
 & 3x^2-2x-1 \quad \textit{Answer.} \\
3x^2\;) & 9x^4-12x^3-2x^2+4x+1 \\
 & 9x^4 \\
6x^2-2x\;) & \quad -12x^3-2x^2+4x+1 \\
 & \quad -12x^3+4x^2 \\
6x^2-4x-1\;) & \qquad -6x^2+4x+1 \\
 & \qquad \underline{-6x^2+4x+1} \\
 & \qquad \cdot \quad \cdot \quad \cdot
\end{array}$$

The same method is used in Arithmetic.

Example i.—*Find* $\sqrt{116964}$.

$$\begin{array}{lrl}
 & & \text{H. T. U.} \\
 & & 3\;4\;2 \quad \textit{Answer.} \\
\text{H.} & & \\
3 &) & 11\,69\,64 \\
 & & 9 \\
\text{H. T.} & & \\
6\,4 &) & 2\,69 \\
 & & 2\,56 \\
\text{H. T. U.} & & \\
6\,8\,2 &) & 13\,64 \\
 & & \underline{13\,64} \\
 & & \cdot \;\; \cdot
\end{array}$$

Notice that:

(i) For every two digits in the given number there is one digit in the square root.

(ii) The divisor 64 is really twice 3 hundreds + 4 tens; i.e. 640, and the divisor 682, twice (3 hundreds + 4 tens) + 2 units.

EXAMPLE ii.—*Find* $\sqrt{678 \cdot 285}$.

(i) Mark the digits off in pairs from the decimal point. Notice that on the extreme left 6 stands alone, but that on the extreme right a nought is added to complete the pair.

(ii) Proceed as in Example i.

(iii) On bringing down 28 and trying 1 in the answer, 521 is obtained for the number to be subtracted. As this is greater than 228, place 0 in the answer *and in the divisor*, and bring down the next two digits.

```
             2  6 · 0  4    Answer.
       2 ) 6 78·28 50
           4
      46 ) 2 78
           2 76
    5204 )    2 28 50
              2 08 16
   10408 )   20 34 00
```

The answer 26·04 is correct to the second decimal place, for the third decimal figure will be found to be less than 5

§5. The Right-angled Triangle.

A knowledge of square root is necessary for the solution of problems referring to the sides of a right-angled triangle.

If c is the hypotenuse of a right-angled triangle, and a and b the remaining sides, then

$$c^2 = a^2 + b^2,$$

from which $$c = \sqrt{a^2 + b^2},$$

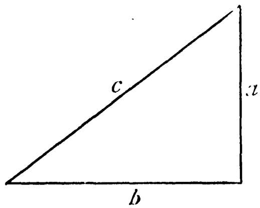


FIG. 7.

i.e. *the hypotenuse is equal to the square root of the sum of the squares of the remaining sides.*

EXAMPLE i.—*If* $a = 3$ *cms. and* $b = 4$ *cms., find* c.

$$c = \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (4^2)} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ cms.}$$

Again, since $$c^2 = a^2 + b^2,$$

$$a^2 = c^2 - b^2 \quad \text{and} \quad b^2 = c^2 - a^2,$$

from which $$a = \sqrt{c^2 - b^2} \quad \text{and} \quad b = \sqrt{c^2 - a^2},$$

i.e. *a side, not the hypotenuse, is equal to the square root of the difference between the squares of the hypotenuse and of the remaining side.*

EXAMPLE ii.—*If* $c = 5$ *cms. and* $b = 4$ *cms., find* a.

$$a = \sqrt{c^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3 \text{ cms.}$$

EXAMPLE iii.—*If* $c = 5$ *ins. and* $a = 3$ *ins., find* b.

$$b = \sqrt{c^2 - a^2} = \sqrt{25} - 9 = \sqrt{16} = 4 \text{ ins.}$$

EXERCISE VII (E)

Find the square roots of the following:

1. $4a^2 + 20ab + 25b^2$.
2. $4a^2 - 20ab + 25b^2$.
3. $16x^2 - 40xy + 25y^2$.
4. $36x^4 - 12x^2 + 1$.
5. $a^4 - 4a^3 + 8a + 4$.
6. $1 - 4y + 6y^2 - 4y^3 + y^4$.
7. $x^4 - 2x^3y + 5x^2y^2 - 4xy^3 + 4y^4$.
8. $9a^4 - 12a^3 + 10a^2 - 4a + 1$.
9. 9801.
10. 15129.
11. 3080·25.

Determine to the third decimal place:

12. $\sqrt{2}$.
13. $\sqrt{3}$.
14. $\sqrt{5}$.
15. $\sqrt{6}$.
16. $\sqrt{7}$.
17. $\sqrt{8{\cdot}263}$.
18. $\sqrt{0{\cdot}03856}$.
19. $\sqrt{231{\cdot}5}$.
20. $\sqrt{1583{\cdot}62}$.
21. Calculate the hypotenuses of the right-angled triangles of which the sides are:

   (i) 5 cms. and 12 cms. (ii) 7 cms. and 24 cms.

   (iii) 40 cms. and 9 cms. (iv) 13 inches and 84 inches.

   (v) 33 ft. and 56 ft. (vi) 35 cms. and 12 cms.

   (vii) 2·4 ins. and 3·6 ins. (Answer to first decimal place.)

22. Find the remaining side of each of the following right-angled triangles:

(i)	Hypotenuse,	13 cms.;	one side,	5 cms.
(ii)	,,	73 cms;	,,	48 cms.
(iii)	,,	117 cms.;	,,	45 cms.
(iv)	,,	29 ins.;	,,	21 ins.
(v)	,,	109 ins.;	,,	91 ins.
(vi)	,,	85 ft.;	,,	36 ft.
(vii)	,,	$3\sqrt{2}$ cms.;	,,	3 cms.
(viii)	,,	30·3 ft.;	,,	12·2 ft.

23. A ladder 40 ft. long is placed against a house so that its upper end just reaches a spout 35 ft. above the ground. How far is the foot of the ladder away from the wall?

24. Calculate to two places of decimals the length of the diagonal of a square of side 8 cms.

25. The adjacent sides of an oblong measure 10 cms. and 15 cms. Find the length of its diagonal.

26. Find, by the use of a right-angled triangle, the radius of a pipe which has a section equal to the sum of the sections of two given circular pipes.

27. The figure represents the internal sections of two pipes of given diameters.

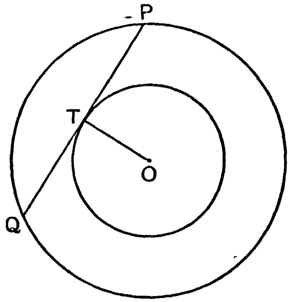


FIG. 8.

Show that PQ represents the diameter of a pipe of which the area of section is the difference between those of the given pipes, and that the area of the ring between the two circumferences is $\pi(PT)^2$.

CHAPTER VIII

RATIO AND PROPORTION

§1. Ratio.

A vulgar fraction, such as $\frac{3}{5}$, is sometimes called a ratio. The ratio $\frac{3}{5}$ (spoken, "as 3 is to 5") is sometimes written in the form $3:5$. The first term, 3, is called the *antecedent*, and the second, 5, the *consequent*.

A ratio represents the *relative* magnitude of quantities of the same kind.

Thus two lines, a and b, when measured by a scale of inches measure 3 inches and 5 inches respectively. Then the ratio of the length of a to that of b is $\frac{3}{5}$.

This is a fixed relation between the lengths of these two lines, no matter by what scale we measure them. If they are measured by a scale which has half inches as units, then a will measure 6 units and b 10 units. The ratio of these lengths is $\frac{6}{10}$, and this is equal to the ratio $\frac{3}{5}$.

Ratios, like vulgar fractions, can be cancelled, or multiplied above and below, by the same number without altering their value.

When the antecedent is *greater* than the consequent, the ratio is said to be of *greater inequality*; when *less*, of *less inequality*.

When comparing ratios it will often be found convenient to work out the quotient of the terms in decimal form.

Exercise VIII (a)

1. Express the following ratios in their simplest form:

$$\frac{10}{12},\ \frac{81}{72},\ \frac{10}{6},\ \frac{1\frac{1}{2}}{2},\ \frac{3\frac{1}{4}}{2\frac{1}{5}},\quad 15:25,\quad 8:12,\quad 3{\cdot}5:4{\cdot}2.$$

2. Draw any two straight lines, measure them in inches and also in centimetres, and write down the ratios of their lengths. Show as clearly as you can that the ratios are approximately equal.

3. What is the ratio of the perimeter of a square to one of its sides?

4. What is the ratio of the area of a square to that of the square of half the side?

5. What is the ratio of the value of a penny to the value of a shilling? Will your result represent the ratio of the weights also?

6. What is the ratio of the value of a florin to the value of a half-a-crown?

   If you have a balance and weights, see if the ratio of their weights is the same as that of their value.

7. What is the ratio of the area of a circle to that of the square on the radius, and also to that of the square on the diameter?

8. A circle is described to touch the sides of a square. What is the ratio of one of the corner areas to the whole square?

9. Arrange the following ratios in order of magnitude:

$$\tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{7}, \tfrac{5}{9}, \tfrac{5}{8}, \tfrac{6}{11}.$$

10. A bottle when filled with water weighs W grams, and when filled with milk M grams. If the empty bottle weighs B grams, what is the ratio of the weight of milk to the weight of an equal volume of water?

§ 2. Proportion.

Proportion is the equality of ratios.

Draw two straight lines of length 3 inches and 6 inches respectively. The ratio of their lengths is $\frac{3}{6}$, or $\frac{1}{2}$. Draw two other lines of length 8 and 16 inches. Their ratio is $\frac{8}{16}$, or again $\frac{1}{2}$. Their ratios are therefore equal. We can write

$$\tfrac{3}{6} = \tfrac{8}{16}.$$

Four numbers so related are called proportionals.

The first and last terms are called extremes, the second and third (6 and 8) means.

The fourth term (16) is called the fourth proportional of 3, 6 and 8.

If any one of the four numbers is unknown, it can be readily calculated.

Suppose that we did not know the length of the first line, but that we knew the lines to be proportional.

Let $x =$ the unknown length.

Then $$\frac{x}{6} = \frac{8}{16},$$

from which $$x = 3.$$

The four quantities need not be of the same kind. E.g. the value of gold is proportional to its weight. That is, the ratio of two weights is equal to the ratio of the corresponding values of the weights.

Thus, if 2 oz. are worth £5, then 6 oz. are worth £15.

The proportion is $$\tfrac{2}{6} = \tfrac{5}{15}.$$

Such a proportion is called a *direct* proportion, because the antecedents of both ratios refer to one quantity. The consequents, likewise, both refer to one other quantity.

The necessity for a distinction will be seen when indirect or inverse proportion is considered.

It is well known that the faster a train travels, the less time it takes to cover a certain distance.

Suppose a train, A, goes at 40 miles an hour, and another, B, at 30 miles an hour. Then to travel a particular distance, say 240 miles, A takes 6 hours and B takes 8 hours.

Now, $$\frac{\text{Speed of A}}{\text{Speed of B}}=\frac{40}{30},$$

and $$\frac{\text{Time of A}}{\text{Time of B}}=\frac{6}{8}.$$

A glance at the ratios shows that they are not directly equal, i.e. when both numerators refer to A and both denominators to B. But if the second ratio be inverted, we have two equal ratios and therefore a proportion.

$$\frac{\text{Speed of A}}{\text{Speed of B}}=\frac{40}{30}=\frac{8}{6}=\frac{\text{Time of B}}{\text{Time of A}},$$

i.e. the ratio of the speeds is equal to the inverse ratio of the times.

This is an *inverse* proportion.

§3. Important Deductions in Proportion.

1. If $\frac{a}{b}=\frac{c}{d}$, then $$\frac{b}{a}=\frac{d}{c}.$$

Divide 1 by each ratio; then $$\frac{1}{\frac{a}{b}}=\frac{1}{\frac{c}{d}},$$

i.e. inverting and multiplying, $$\frac{b}{a}=\frac{d}{c}.$$

2. If $\frac{a}{b}=\frac{c}{d}$, then $$ad=bc.$$

Multiply both sides by bd (product of the denominators):

$$\frac{a}{\cancel{b}}\times \cancel{b}d=\frac{c}{\cancel{d}}\times b\cancel{d},\quad \text{i.e.,}\quad ad=bc.$$

In words, the product of the extremes is equal to the product of the means.

Special case. When b and c are equal, the product of the extremes is equal to the square of one of the means. E.g. if $\frac{a}{b}=\frac{b}{c}$, then $ac=b^2$.

c is called the *third proportional* of a and b, and b the *geometric mean* of a and c.

3. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$,

Since $\frac{a}{b}=\frac{c}{d}$, $ad=bc$.

Divide both sides by cd; then $\frac{a\cancel{d}}{c\cancel{d}}=\frac{b\cancel{c}}{\cancel{c}d}$, $\therefore \frac{a}{c}=\frac{b}{d}$.

4. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Add 1 to each side; then $\frac{a}{b}+1=\frac{c}{d}+1$,

from which $\frac{a+b}{b}=\frac{c+d}{d}$.

5. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

Subtract 1 from each side; then $\frac{a}{b}-1=\frac{c}{d}-1$,

from which $\frac{a-b}{b}=\frac{c-d}{d}$.

6. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

Divide result 5 by result 4; b cancels on one side, and d on the other.

$$\therefore \frac{a-b}{a+b}=\frac{c-d}{c+d}.$$

Exercise VIII (b).

Find x in the following proportions:

1. $\frac{x}{9}=\frac{25}{27}$. 2. $\frac{9}{x}=\frac{15}{4}$. 3. $\frac{5}{8}=\frac{x}{14}$. 4. $\frac{5}{8}=\frac{14}{x}$.

5. $\frac{x}{a}=\frac{b}{c}$. 6. $\frac{a}{x}=\frac{b}{c}$. 7. $\frac{a}{b}=\frac{x}{a}$. 8. $\frac{a}{b}=\frac{a}{x}$.

9. Examine the following ratios, and determine whether they are directly or inversely equal:

(i) $\frac{6}{10}$ and $\frac{9}{15}$. (ii) $\frac{8}{12}$ and $\frac{36}{24}$.

(iii) $\frac{12}{10}$ and $\frac{2\cdot4}{2}$. (iv) $\frac{10}{35}$ and $\frac{28}{8}$.

10. The following ratios are equal in pairs. Find the missing terms.

(i) $\frac{?}{5}$ and $\frac{6}{10}$. (ii) $\frac{?}{5}$ and $\frac{10}{6}$.

(iii) $\frac{4}{?}$ and $\frac{12}{36}$. (iv) $\frac{9}{2}$ and $\frac{?}{5}$.

11. Say whether the following are direct or inverse proportions:

(i) Circumference of a wheel A, is 10 feet.

Number of turns in a fixed distance = 528.

Circumference of a wheel B, is 12 feet.

Number of turns in the same distance = 440.

(ii) Circumference of a wheel A, is 10 feet.

Distance covered in a number of turns = 500 ft.

Circumference of a wheel B, is 12 feet.

Distance covered in the same number of turns = 600 ft.

12. An experiment showed that the weights of pieces of the same sheet of drawing paper were directly proportional to the areas of the surfaces. A portion having the shape of the map of Ireland weighed 3·052 grams, and another piece in the shape of a rectangle, the sides of which represented 235·2 and 292·8 miles on the same scale as the map, weighed 6·402 grams. Calculate the area of Ireland.

13. Draw a circle, and by drawing radii at angles 30°, 45°, 60°, 72,° 90°, etc., divide the circle into sectors.

By means of thread, transparent paper or other means, measure the arcs of these sectors.

Form a table.

Angle, -	30°	45°	60°	...	270°	360°
Arc, - -						

(i) Compare the ratio of the angles with the ratio of the arcs.

(ii) What fraction of the whole circumference is each arc? How can this fraction be determined from the angle?

(iii) If the radius of the circle is R and the angle of the sector is $x°$, what is the length of the arc?

14. The radius of a circular arc is 5 cms. If the arc subtends an angle of 30° at the centre, calculate its length.

15. How would you divide a circle into twelve sectors of equal area? What reasons have you for saying the sectors are equal?

16. Compare the area of a sector, the angle of which is 60°, with the area of the whole circle of which it is a part.

17. A sector has a radius R and an angle $x°$. What fraction of the area of the whole circle is the area of this sector?

Find the formula for the area of the sector.

18. Find the area of each sector of Exercise 13.

19. Take a point P within a circle and through P draw a number of chords, and a diameter of the circle.

Measure the segments of the chords and form a table, thus

	Segments on left of diameter.	Segments on right of diameter.
1st Chord,	0·4 in.	1·8 ins.
2nd "	0·6 "	1·2 "
Etc.		

Examine ratios of these numbers, and draw your conclusions.

20. In an experiment in which different weights of hot and cold water were mixed together and the charge in temperature noted, the following numbers were obtained:

	Weight of cold water.	Weight of warm water.	Rise in temp. of cold water.	Fall in temp. of hot water.
Exper. 1.	50 grms.	50 grms.	24°	23·5°
" 2.	25 "	50 "	29	15
" 3.	75 "	50 "	20	29
" 4.	50 "	75 "	26	20
" 5.	50 "	25 "	16	31

Examine the following ratios:

$$\frac{\text{Wt. of cold water}}{\text{Wt. of warm water}}, \quad \frac{\text{Rise in temperature of the cold water}}{\text{Fall in temperature of the hot water}},$$

and determine whether the ratios are, approximately directly or inversely equal.

Employ what you have learnt to find the temperature of a mixture of 100 grams of water at 20° with 80 grams at 60°

21. The following numbers were obtained when experimenting to find how the weight, necessary to balance a weight of 200 grams on the other side of the fulcrum of a lever and at a fixed distance 6 cms. from it, changed when its distance from the fulcrum was altered:

Left side.		Right side.	
Weight.	Distance.	Weight.	Distance.
200 grms.	6 cms.	50 grms.	24 cms.
" "	"	60 "	20 "
" "	"	80 "	15 "
" "	"	100 "	12 "
" "	"	120 "	10 "
" "	"	150 "	8 "
" "	"	200 "	6 "
" "	"	300 "	4 "
" "	"	400 "	3 "

Examine the ratio of the weights and the ratio of the distances on the right, and draw your conclusions.

Examine also the ratio of a weight on the right to the weight on the left and the ratio of the corresponding distances. Draw your conclusions.

22. In an experiment on the inclined plane, the following numbers were obtained:

Weight raised.	Effort applied.	Length of plane.	Height of upper end plane.
278 grams.	68 grams.	52 cms.	13·5 cms.
278 "	90 "	52 "	17 "
278 "	105 "	52 "	20 "
278 "	140 "	52 "	26 "
278 "	161 "	52 "	30 "

Compare the ratios $\dfrac{\text{Effort}}{\text{Weight raised}}$ and $\dfrac{\text{Height}}{\text{Length}}$ of plane.

State your conclusion in algebraic form. What effort will be necessary to raise the weight when the height of the upper end of the plane is 24 cms.?

23. If x c.cs. of an acid solution are required to neutralise y c.cs. of an alkali solution, express their relative strength as a ratio. If it takes z c.cs. of another acid solution to neutralise y c.cs. of the same alkali solution, express the strengths of the acid solutions as a ratio.

24. Find the arc between the ends of two radii of a circle, which make an angle of 150° and are 3 inches in length.

25. The arc of a sector of a circle of 5 ins. radius measures 5 ins.; calculate the angle between its bounding radii. This angle is double any angle striding this arc, but with its vertex anywhere on the remaining part of the circumference of the circle. What is the angle in this case?

26. How many minute spaces does the large hand of a clock gain on the small hand in x minutes?

27. A ball of copper weighs A grams in air and W grams when totally submerged in water. What is the difference in weight?

This difference is exactly equal to the weight of the liquid displaced by the copper. Express as a ratio, the weight of the ball compared with the weight of an equal volume of water.

28. A metal ball weighs A grams in air, W grams in water (totally submerged) and T grams when totally submerged in turpentine.

Express as a ratio the weight of turpentine compared with the weight of an equal volume of water.

29. A bottle weighs B grams when empty and W grams when filled with water. A cylinder of copper weighing C grams is lowered into the bottle of water and the bottle filled again to the brim should too much have run out. If bottle and contents now weigh G grams, what weight of water does the copper cylinder displace?

30. Write down as many new proportions as you can from:

$$\frac{3x}{2y}=\frac{5a}{8b}.$$

31. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{a}=\frac{c+d}{a}$, and that $\frac{a-b}{a}=\frac{c-d}{c}$.

32. The rims of two wheels are touching. If the diameter of one wheel is 12 inches and of the other 3 inches, find how many times the smaller will turn when the larger makes one revolution. How do the revolutions depend upon the diameters?

33. Two wheels are geared together by means of a belt. If there is no slipping, show that the number of revolutions made by the wheels in the same time is inversely proportional to the diameters of the wheels.

34. A wheel having 12 teeth is geared to a wheel with 36 teeth. What will be the ratio of their revolutions in the same time?

§ 4. Similar Triangles.

Draw a straight line AB, say 3 inches long, and from the end A draw another straight line AX, making an angle with AB.

Along AX, with a pair of compasses step off, say, 5 equal lengths.

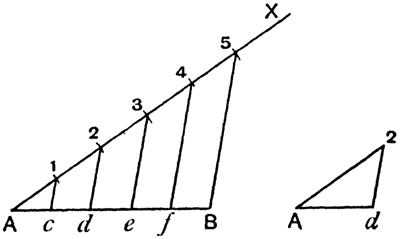


FIG. 1.

Draw a straight line from the point marked 5 to the end B, and from the remaining marked points draw straight lines parallel to 5B to cut AB. Mark the points of intersection *c*, *d*, *e*, *f*.

Now measure the lengths A*c*, *cd*, *de*, *ef*, *f*B.

What is your conclusion?

Still using compasses, compare the lengths of *c*1, *d*2, *e*3, *f*4, B5.

Now look at the triangles AB5 and, say, A*d*2.

They have the same angle A; the angle at *d* is equal to the angle at B, and the angle at 2 is equal to the angle at 5. They have the same shape.

One is, however, bigger than the other, but there is a very special relation between their sides.

Find the following ratios:

$$\frac{\text{side (AB)}}{\text{side (A}d\text{)}}, \quad \frac{\text{side (A5)}}{\text{side (A2)}}, \quad \frac{\text{side (B5)}}{\text{side (}d2\text{)}}.$$

Your conclusion is, that all these ratios are equal.

Such triangles are said to be similar.

Notice that the terms of each ratio are opposite equal angles.

Sides opposite equal angles are called corresponding sides. In similar triangles the ratios of pairs of corresponding sides are equal, i.e.

$$\frac{\mathrm{AB}}{\mathrm{A}d}=\frac{\mathrm{A5}}{\mathrm{A2}}=\frac{\mathrm{B5}}{d2}.$$

The triangles need not, of course, be one inside the other, but may be quite apart, as shown in the figure.

The point to remember is, that they must be equiangular.

The ratios may be written so that the terms of a ratio refer to the same triangle.

Thus: since
$$\frac{\mathrm{AB}}{\mathrm{A}d}=\frac{\mathrm{A5}}{\mathrm{A2}},$$
$$\therefore \frac{\mathrm{AB}}{\mathrm{A5}}=\frac{\mathrm{A}d}{\mathrm{A2}}.$$

Observe that the sides forming the terms of these ratios contain equal angles.

§ **5. Applications.**

(1) To find the fourth proportional to three given straight lines, a, b and c.

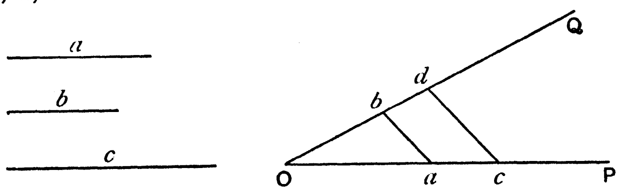


Fig. 2.

From a point O, draw two straight lines, OP and OQ, at a convenient angle.

Along OP mark off a length Oa equal to a.

" OQ " " Ob " b.

" OP " " Oc " c.

Join points a and b, and through c draw cd parallel to ab; then Od is the fourth proportional required. Call it d.

For since ab is parallel to cd,

$$\frac{\text{O}a}{\text{O}b}=\frac{\text{O}c}{\text{O}d},$$

$$\text{i.e.}\quad \frac{a}{b}=\frac{c}{d}.$$

Observe that (i) if b is equal to unity, a is equal to $\frac{c}{d}$, i.e. a is the quotient of c divided by d.

(ii) If a is unity, then $\frac{1}{b}=\frac{c}{d}$ or $d=bc$, i.e. d is the product of b and c.

These facts can be used to determine straight lines which represent the product and quotient of two other straight lines, when the length of the unit is given.

All that is required is a little foresight in determining upon which line a particular term shall be measured, and which points shall be joined.

Thus, in fig. 3, in which Ou represents the unit, if u and a are joined and the parallel drawn through b, then Ox is the product of a and b.

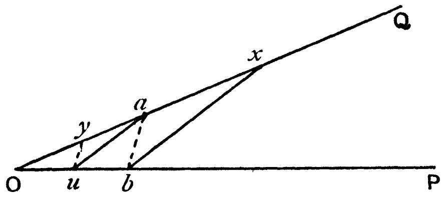


FIG. 3.

If a and b are joined and the parallel drawn through u, then Oy is the quotient of a by b.

If the quotient of b by a is required, u should be taken on the same line as a.

In the figure, notice that Oy is greater than Ou, which is as expected, for a is greater than b, and the quotient therefore greater than unity.

(2) The sides AB, BC and AC of a triangle ABC (fig. 4) are 3·5, 3·0 and 4·7 cms. respectively.

A line DE is drawn parallel to the base BC from a point D in AB, 2 cms. from A. Find the length of DE and of AE.

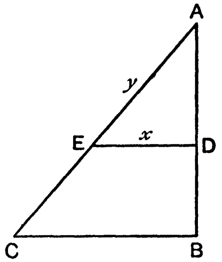


FIG. 4.

Let DE $= x$ and AE $= y$.

Since triangles ADE and ABC are similar,

$$\therefore \frac{x}{3} = \frac{2}{3 \cdot 5} \quad \text{and} \quad \frac{y}{4 \cdot 7} = \frac{2}{3 \cdot 5}.$$

From these simple equations x and y are readily calculated. Find them, and check your result by measurement.

(3) AB is a diameter of a circle, and it intersects a chord CD at right angles.

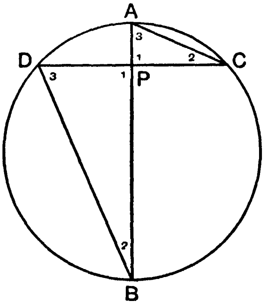


FIG. 5.

Since the right-angled triangles APC and BPD are similar,

$$\frac{AP}{PC} = \frac{PD}{PB},$$

$$PC \times PD = AP \times PB;$$

and since PC $=$ PD,

$$(PC)^2 = AP \times PB.$$

That is, the square of half the chord is equal to the product of the segments of the diameter.

(4) PT is a tangent to a circle from a point P, in the diameter BA, produced.

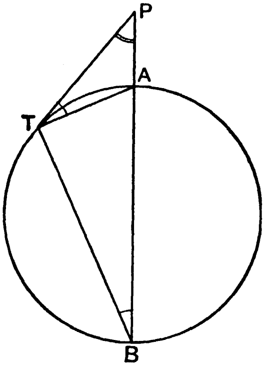


FIG. 6.

The triangles PAT and PBT are similar;

$$\therefore \frac{PA}{PT}=\frac{PT}{PB},$$

from which $$PT^2=PA\times PB.$$

That is, the square of the tangent is equal to the product of the distances of its intersection with a diameter from the ends of the diameter.

These relations are of wide application.

EXERCISE VIII (C)

1. In fig. 4, show that $\frac{AE}{EC}=\frac{AD}{DB}$.

2. Draw any triangle, and from any point on one side draw a straight line parallel to the base to intersect the remaining side. Letter the figure, and write down all the pairs of equal ratios you can find. Verify your statements by careful measurements.

3. Draw two straight lines x and y, 1·8 inches and 3 cms. long respectively. Taking the unit to be a straight line an inch long, find lines to represent

   (i) xy. (ii) $\frac{x}{y}$. (iii) $\frac{y}{x}$. (iv) x^2. (v) y^2.

   Measure the lines in inches, and check the results.

4. Using the lines in Ex. 3 for x and y, find a line representing $\frac{x+3y}{7}$.

5. Taking an inch as the unit, find straight lines to represent

$$\text{(i) } 2{\cdot}3 \times 0{\cdot}8. \qquad \text{(ii) } \frac{2{\cdot}3}{0{\cdot}8}. \qquad \text{(iii) } \frac{0{\cdot}8}{2{\cdot}3}.$$

Check by measurement.

6. When the shadow of a vertical stick 6 feet long measures 8 feet, that of a building measures 75 feet. Find the height of the building.

7. If, in fig. 5, AB is 5 inches and AP 1 inch, calculate PC.

8. If, in fig. 5, PC is 6 cms. and AP 4 cms., find AB.

9. In fig. 6, from T draw the chord at right angles to AB, and let it intersect AB at Q. Show that

$$PA \times QB = AQ \times PB.$$

CHAPTER IX

SPECIAL RATIOS, TRIGONOMETRY

§ **1.** The ratios of the sides of a right-angled triangle are of special importance in a branch of Mathematics, called Trigonometry.

Draw a right-angled triangle ABC, having the angle A, say 40° (fig. 1).

The side AB is called the hypotenuse, and regarding the remaining sides from the angle A, BC is the opposite side and AC the adjacent side.

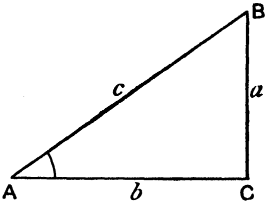


FIG. 1.

The ratio, $\frac{\text{opposite side}}{\text{hypotenuse}}$, is called the **sine** of the angle from which the triangle is regarded.

E.g. sine A (usually written sin A) $= \frac{BC}{AB}$ or $\frac{a}{c}$.

The ratio, $\frac{\text{adjacent side}}{\text{hypotenuse}}$, is called the **cosine** of the angle.

E.g. cosine A (briefly written cos A) $= \frac{AC}{AB}$ or $\frac{b}{c}$.

The ratio, $\frac{\text{opposite side}}{\text{adjacent side}}$, is called the **tangent** of the angle.

E.g. tangent A (briefly written tan A) $= \frac{BC}{AC}$ or $\frac{a}{b}$.

You will notice that the sides are lettered according to the angle to which they are opposite, but in small letters.

Exercise IX (a)

1. Measuring the sides of the triangle constructed, find sin 40°, cos 40° and tan 40°.
2. The remaining acute angle is 50°. Regarding the sides from this angle, find sin 50°, cos 50° and tan 50°.
3. What conclusion do you draw concerning the ratios of an angle and the ratios of its complement?
4. Construct appropriate right-angled triangles, take measurements, and find the trigonometrical ratios of 30°, 45° and 60°.

Remember that the terms sine, cosine and tangent merely denote ratios, and may be regarded as algebraic numbers.

The ratios for angles up to 90° will be found in the book of tables referred to in Chapter XIV, § 13.

§ 2. Simple Applications of the Trigonometrical Ratios.

ABC is a right-angled triangle, with $B\hat{A}C$ 35° and AC(b) 2·5 cms. Find the remaining sides.

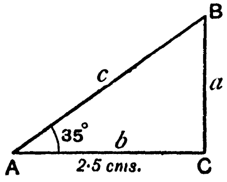


Fig. 2.

$$\frac{a}{b} = \tan 35°, \quad \therefore \quad a = b \tan 35° = 2{\cdot}5 \times {\cdot}7002 = 1{\cdot}75 \text{ cms.}$$

$$\cos 35° = \frac{b}{c}, \quad \therefore \; c = \frac{b}{\cos 35°} = \frac{2{\cdot}5}{{\cdot}8192} = 3{\cdot}06 \text{ cms.}$$

EXERCISE IX (B)

Find the remaining sides of the following right-angled triangles, C being the right angle:

1. A = 30°, a = 3 inches.
2. B = 50°, a = 3·2 cms.
3. A = 30°, c = 3 inches.
4. B = 50°, c = 3·2 cms.
5. A = 85°, b = 4 inches.
6. A = 40°, c = 2·5 inches.
7. Find the area of each of the above triangles.
8. If, in fig. 1, BC represents a vertical object and AC a horizontal line, ∠A is called the angle of elevation of the top of the object. When AC and ∠A are known, BC can be calculated.

   The elevation of the top of a tower at a point 300 feet from its foot is 40°. Calculate the height of the tower.
9. If the sun-shadow cast by a vertical pole 6 feet high is 8 feet, calculate the altitude of the sun.
10. The angle that BA makes with the horizontal through B (fig. 1) is called the angle of depression. From the property of parallels, the angle of depression of BA is equal to the angle of elevation of AB.

    From the top of a cliff 500 feet high, the angles of depression of two boats at sea are observed to be 45° and 30° respectively; the line joining the boats points directly to the foot of the cliff. Find the distance between the boats.

§ 3. The Length of any Parallel of Latitude.

We shall assume the earth to be a sphere.

Let ∠POW = the angle of latitude (L) of any place P (fig. 3).

PQ is the radius of the line of latitude of P, and OP the radius of the earth.

Then ∠QPO = alt. ∠POW = L.

Hence, regarding

△PQO from ∠QPO,

$$\frac{PQ}{OP} = \cos L;$$

$$\therefore\ PQ = OP \cdot \cos L = R \cos L.$$

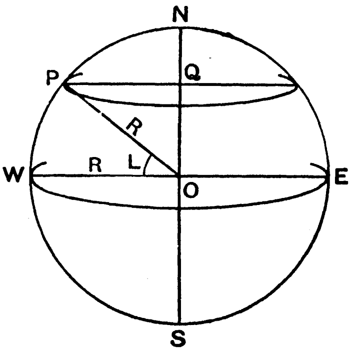


FIG. 3.

Now, since the line of latitude of P is the circumference of the circle of which PQ is the radius, we have:

Length of parallel of latitude $= 2\pi \times PQ$

$$= 2\pi R \cos L.$$

The portion of this line of latitude lying between two lines of longitude, one degree apart, is $\frac{2\pi R \cos L}{360}$, since the complete cycle is 360°.

At the Equator, $L = 0°$, and since $\cos 0 = 1$, this formula becomes

$$\frac{2\pi R}{360}.$$

EXAMPLE.—Taking the earth to be a sphere of 4000 miles radius, find the length of the line of latitude 51° N. between any two lines of longitude one degree apart.

From the tables, $\cos 51° = \cdot 6923$.

$$\text{Required answer} = \frac{2\pi \times 4000 \times \cdot 6923}{360} = 48\cdot 38 \text{ miles.}$$

It will be readily understood that the portion of a line of longitude lying between two lines of latitude one degree apart is the same for all longitudes and latitudes, and is equal to

$$\frac{2\pi R}{360} = \frac{\pi R}{180} = 69\cdot 8 \text{ miles.}$$

§ **4.** The following relation between the sides and angles of a triangle is important, and very useful.

Let ABC (fig. 4) be the triangle considered.

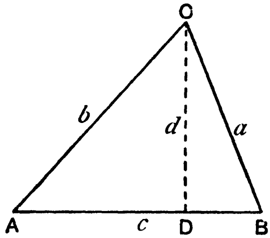


FIG. 4.

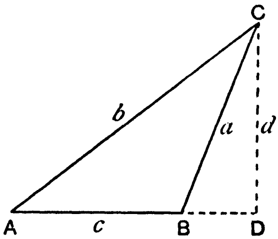


FIG. 5.

From C draw CD perpendicular to AB.

Let $CD = d$; then, from the right-angled $\triangle$ ADC,

$$\frac{d}{b} = \sin A;$$

$$\therefore \; d = b \sin A. \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

From the right-angled $\triangle$ BDC,

$$\frac{d}{a} = \sin B;$$

$$\therefore\ d = a \sin B. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) and (ii) it follows that

$$a \sin B = b \sin A,$$

and
$$\therefore\ \frac{a}{b} = \frac{\sin A}{\sin B}.$$

If the perpendicular is drawn from B to AC or from A to CB, it can be shown that $\frac{a}{c} = \frac{\sin A}{\sin C}$.

In words, **the ratio of any two sides is equal to the ratio of the sines of the opposite angles.**

If one of the angles, say B, is obtuse (fig. 5), the above reasoning still holds, if we substitute the supplement* of B for B. Later, you will learn that the sine of the supplement of an angle is equal to the sine of that angle.

§ 5. Application.

When given two angles and one side of a triangle, the remaining parts can be calculated.

EXAMPLE.—To find the position of an inaccessible object C, the following measurements were made at a base line AB.

AB = 120 yds., $\angle$ BAC = 35°, $\angle$ ABC = 60°.

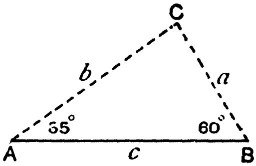


FIG. 6.

The angle

ACB = 180° − (35° + 60) = 85°.

To find AC and BC, we have:

$$\frac{a}{c} = \frac{\sin 35°}{\sin 85°};$$

$$\therefore\ a = c\frac{\sin 35°}{\sin 85°}$$

$$= 120 \times \frac{0{\cdot}5736}{0{\cdot}9962} \text{ yds.}$$

$$= 69{\cdot}1 \text{ yds.}$$

From the relation $\frac{b}{c} = \frac{\sin B}{\sin C}$, b can be found in like manner.

* The supplement of an angle is the amount by which it is less than a straight angle (180°).

Exercise IX (c)

1. Complete the foregoing application, and find also the perpendicular distance from C to the base line.

2. Show that in any triangle:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Calculate the remaining sides of the following triangles:

3. $A=60°$, $B=80°$, $a=2$ cms.

4. $A=60°$, $B=80°$, $b=2$ cms.

5. $A=60°$, $B=80°$, $c=2$ cms.

6. $B=50°$, $C=40°$, $a=10$ cms.

7. $A=100°$, $C=30°$, $b=6$ inches.

8. The directions of an object make angles of 60° and 50° with the directions of a base line 1000 yds. long, when viewed from each end of the base line. Find the position of the object.

9. Making use of fig. 5, show that the area of a triangle is

$$\frac{bc}{2}\sin A \quad \text{or} \quad \frac{ab}{2}\sin C \quad \text{or} \quad \frac{ac}{2}\sin B.$$

10. Calculate the areas of the triangle in Exercises 3 to 7.

11. Assuming the earth to be a sphere of 8000 miles diameter, what is the circumference of the circle of latitude 52°?

    The earth makes one revolution in 24 hours (approximately); what is the speed at latitude 52° in miles per hour?

12. Explain, with a diagram, how you would find the height of a tree, if you have a set-square whose angles are 30° and 60°.

13. A tower, 30 feet high, is surmounted by a vertical flagstaff 34 feet long. At a point P in the horizontal plane through the foot of the tower the flagstaff and the tower subtend equal angles; what is the distance of P from the tower?

14. A right-angled triangular field has one side 150 yds. long, the angle opposite it being 58°. Find the hypotenuse.

15. South America has roughly the shape of two triangles with a common base, as shown on the figure.

From the data given, calculate the approximate area of the country.

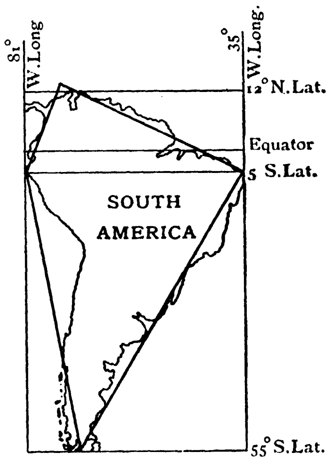


FIG. 7.

16. Refer to your atlas and you will see that India also has the shape of two triangles placed as in fig. 7. In this case the base is on the 25° N. latitude line, the ends being respectively at 67° and 93° E. longitude, and the northern and southern vertices respectively at 34° and 8° N. latitude.

Determine the approximate area of the country.

17. Find the distance between New York (40° N., 74° W.) and Madrid (40° N., $3\frac{1}{2}$° W.).

18. A ship sailing from Portsmouth ($53\frac{1}{2}$° N., 1° W.) to New York (40° N., 74° W.) sails along the meridian until a little south of latitude 50° N. is reached, and then sails west until its longitude is 44° W., after which it sails south until its latitude is 40° N., when it resumes its westerly course for the remainder of the voyage. Calculate the length of the voyage.

19. The figure represents two pulleys connected by a taut belt.

The diameters of the pulleys are 3 ft. and 2 ft. respectively, and their centres are 5 ft. apart. Find by trigonometry:

(i) The angle TOQ.
(ii) The length of TR.
(iii) The reflex angle TOT_1.
(iv) The angle RQR_1.
(v) The length of the belt.

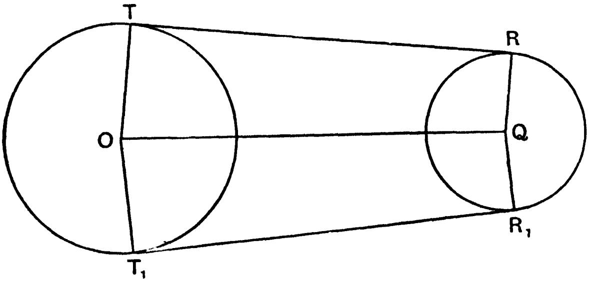


FIG. 8.

Check your result by drawing and measurement.

If the larger pulley makes 100 revs. per minute, find:

(i) The speed of the other pulley.
(ii) The speed of the belt in feet per second.

20. A disc of diameter 6 inches is hung against a wall by a string which passes over a nail 5 inches above the centre of the disc, and round a part of the rim of the disc.

Sketch the arrangement, and neglecting the thickness of the nail, calculate the length of the string.

REVISION EXERCISE I

1. Simplify:

(i) $12-7+(-8)-4-(-3)$.
(ii) $2\times -3+(5)^2-4\div -2-3(6-4)-(-3)^2$.
(iii) $6(a+b)-2(a+b)+4(a+b)-(a+b)$.
(iv) $5a(a-b)+2(x+y)+2a(a-b)-4(x+y)-3a(a-b)$.

2. (i) Subtract 12 from -8. (ii) From -7 subtract -13.
(iii) Subtract $2x^2-5x-8$ from $-6x^2+3x-5$.
(iv) From $-3(2a-6b)$ take $-7(a-3b)$.

3. Calculate the position of the point midway between the points situated at distances 3·2 and $-1{\cdot}4$ inches respectively from zero.

4. Find the value of :

   (i) $(3x+2y)a-b(2x-3y)$, when $ax=3$, $ay=-2$, $bx=1$, and $by=-1$.

   (ii) $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, when $a=4$, $b=-5$, $c=1$, and when $a=2$, $b=-2$, $c=-4$.

5. A kite has the shape of an equilateral triangle with a semicircle on one side. If s is the length of the side of the triangle, find (i) the perimeter, (ii) the area of one face of the kite.

6. If $y=367+2{\cdot}35(x-36)$, find the difference between the values of y when $x=52$ and $x=12$.

7. Show that $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$. From this identity write down the square of $(a+2x-3y)$.

8. For what value of x does $\dfrac{8x}{3+x}$ equal (i) 7, (ii) 9? Can $\dfrac{8x}{3+x}$ have the value 8?

9. Solve (i) $9(x+3)^2+15(x+5)^2=24(x+8)^2$.

   (ii) $\dfrac{5(7x+6)}{23}+\dfrac{9x-1}{2}=50+\dfrac{2x-3}{3}$.

10. If $\dfrac{\mathrm{M}}{\mathrm{N}}=\dfrac{1+x}{1-x}$, show that $x=\dfrac{\mathrm{M}-\mathrm{N}}{\mathrm{M}+\mathrm{N}}$.

11. The perimeter of a 60°, 30°, right-angled triangle is 30 inches; find the length of each side, and the area of the triangle.

12. Find (i) $\dfrac{\sin 2x-\cos 2x}{2}$, when $x=30°$, and when $x=45°$.

    (ii) $\dfrac{\sin\dfrac{\mathrm{D}+\mathrm{A}}{2}}{\sin\dfrac{\mathrm{A}}{2}}$, when $\mathrm{D}=48°$ and $\mathrm{A}=60°$.

13. Use fig. 4, page 88, to show that $c=a\cos\mathrm{B}+b\cos\mathrm{A}$. Find similar equations for a and b also.

CHAPTER X

GRAPHS

§ 1. Graphs of Simple Expressions.

Previous work (p. 46) suggests that in representing values graphically, not only should positive values be shown, but negative values also.

If, in fig. 22, Chapter IV, the horizontal line be extended to the left, and measurements made below the horizontal line, negative values of a and of $4a$ can be represented.

The usual method is to draw two straight lines X′OX and Y′OY at right angles, as in fig. 1. These lines are called *axes*, and their intersection, the *origin*. Values of a are measured along, or parallel to, the horizontal axis, and values of $4a$ along, or parallel to, the vertical axis. E.g. when a is -2, $4a$ is -8.

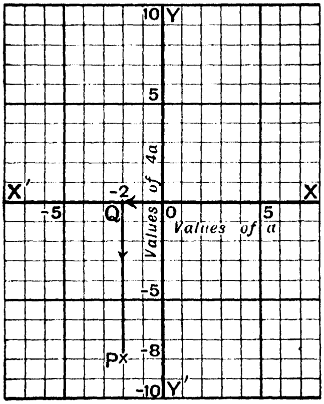


FIG. 1.

These values are represented as follows:

Move from the origin O to -2, on the left, and then move downwards and parallel to OY′, through a distance of -8, as shown by the scale on the vertical axis. Call the final position P. Then P indicates the two values -2 and -8. The distances OQ and QP are called the co-ordinates of P, OQ being named the abscissa (plural, abscissae) and QP the ordinate. In stating co-ordinates it is usual to give the abscissa first; thus the co-ordinates of P are -2, -8.

You must not conclude that ordinates are always negative when the corresponding abscissae are negative.

It is common practice to use x as the symbol. Our object now is to examine the changes in the value of various expressions containing x when the value of x is changed.

Take the simple expression $2x$, and tabulate, as below, its value when x is given the various values shown.

Value of x	-4	-3	-2	-1	$= x =$	0	1	2	3	4	5
Value of $2x$	-8	-6	-4	-2	$= 2x =$	0	2	4	6	8	10

Examining these numbers, it is seen that:

(i) The values of x increase by equal amounts, viz. 1.

(ii) The values of $2x$ increase by equal amounts, viz. 2.

It is clear that when equal changes are made in the value of x, the corresponding changes in $2x$ also are equal, but not necessarily equal to the changes in x.

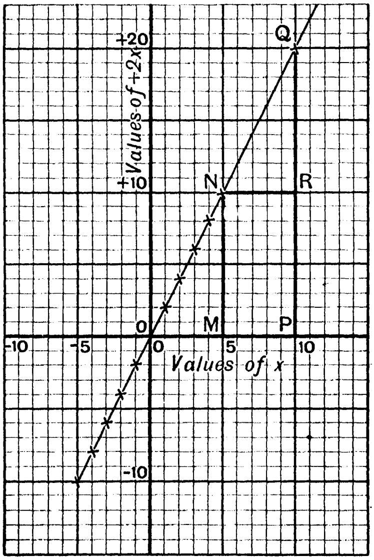


FIG. 2.

Thus, when x changes from 2 to 3, an increase of 1,
$2x$ changes from 4 to 6, an increase of 2,
and when x changes from -4 to -3, an increase of 1,
$2x$ changes from -8 to -6, an increase of 2.

In this case the change in the expression is twice that in x.

It is observed also, that:

(i) When x is $+$, $2x$ is $+$.
(ii) When x is $-$, $2x$ is $-$.
(iii) When x is 0, $2x$ is 0.

Now plot the values, as in fig. 2, and join the points by straight lines. What do you find?

It is not difficult to prove that all the points are in one straight line.

The *Graph* of $2x$ is thus a *straight line.*

Produce the graph in both directions. Take values of x for which you have not calculated the values of $2x$, say $x=\frac{1}{2}$, $-\frac{1}{2}$, -5, etc., and from the graph read off the corresponding values of $2x$. Check these values by actual calculation, and you will find that the graph gives correct results.

Observe further that:

(i) The graph could have been drawn, if two points only had been plotted.

(ii) The graph passes through the origin (0, 0).

(iii) Regarded from the origin towards the right, the graph has an up gradient.

(iv) The ratio of the length of any ordinate to its horizontal distance from the point at which the graph cuts the axis of x, is constant.

$$\text{E.g.}\quad \frac{PQ}{OP}=\frac{MN}{OM}.$$

(v) The ratios referred to in (iv) are each equal to the ratio $\frac{RQ}{NR}$, where NR is parallel to the axis of x, i.e. to the ratio

$$\frac{\text{Difference between any two ordinates}}{\text{Difference between the two corresponding abscissae}}.$$

The ratios named in (iv) and (v) are very important, for they measure the gradient of the graph.

You have probably recognised these ratios as the *tangent of the angle the graph makes with the axis of x.*

Thus:
$$\frac{PQ}{OP}=\tan\angle QOP.$$

In this case $\frac{PQ}{OP}=\frac{20}{10}=2$, hence $\tan\angle QOP=2$.

The graph being a straight line, its gradient is, of course, constant throughout its length.

§2. It is now proposed to determine upon what the gradient depends.

Examine in the same manner the following expressions, namely, $3x$, $4x$ and $\frac{1}{2}x$. Draw the graph of each expression on the same axes as those used for the graph of $2x$ (fig. 3).

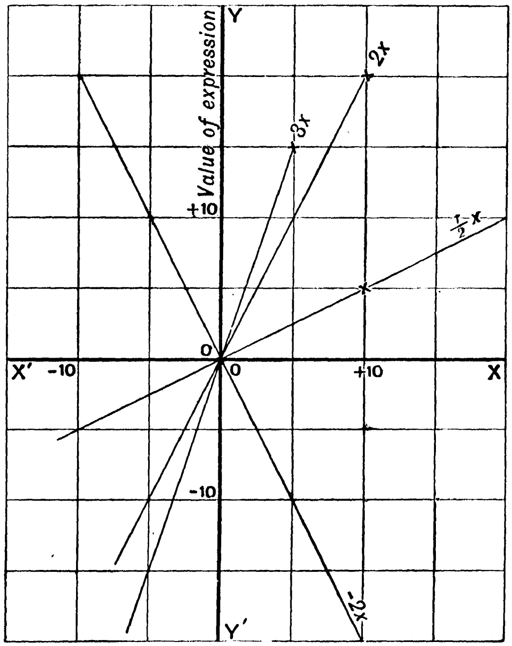


Fig. 3.

Compare the graphs, and observe that:

(i) All pass through the origin.

(ii) All have up gradients.

(iii) The greater the coefficient of x, the greater is the gradient. It is evident that since the expressions differ only in coefficients, the gradient depends upon the coefficient, and may be said to be equal to it. From a table of tangents, find the angle the graph makes with the axis of x in each case. Verify by measurement.*

*The scales of both axes must be the same, otherwise the measured angle will not agree with that given in the table of tangents.

Now plot the graph of $-2x$, and contrast it with that of $+2x$ (fig. 3).

It will be at once observed that the graph of $-2x$ has a down gradient, and you have doubtless concluded that this change in the kind of gradient is due to the change in sign of the coefficient.

A positive coefficient of x gives an up gradient; a negative coefficient, a down gradient.

§ **3.** In order to examine the effect upon the graph, of adding to, or subtracting from $2x$ a constant number, say 3, plot the graphs of $2x+3$ and $2x-3$, and contrast them with the graph of $2x$ (fig. 4). It will be observed that:

(i) All the graphs have the same gradient.

(ii) The graph of $2x+3$ cuts the vertical axis at a distance 3 above the origin.

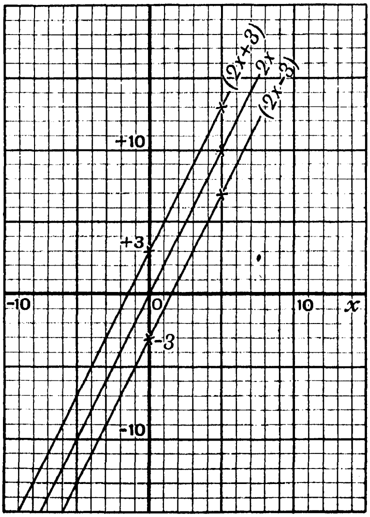


FIG. 4.

(iii) The graph of $2x-3$ cuts the vertical axis at a distance 3 below the origin.

Similarly, examine the graphs of $-2x+3$ and $-2x-3$.

Consider now the point at which, say, the graph of $2x-3$ cuts **the axis of x.**

At this point the value of the expression is 0, and the corresponding value of x is seen to be $1\frac{1}{2}$. This value of x may be obtained by solving the very simple equation

$$2x - 3 = 0.$$

We have, then, a means of determining the point of intersection of the graph with the axis of x.

§ 4. **Summary.**

(i) The graph of an expression of the type $ax + b$, in which a and b are constant numbers, is a straight line.

(ii) The gradient of the graph depends upon the coefficient of x, and is "up" if the coefficient is positive, "down" if negative.

(iii) The position of the graph with respect to the origin depends upon the added constant. If positive, the graph cuts the vertical axis above the origin; if negative, below the origin.

It is usual to call the value of the expression y, and the axis upon which it is shown, the axis of y.

The equation $$y = ax + b$$
is then called a linear equation, for its graph is a straight line. Observe that it contains the first power only of x.

y is said to be a linear function of x.

Notice that when a value is given to x, the value of y becomes definite.

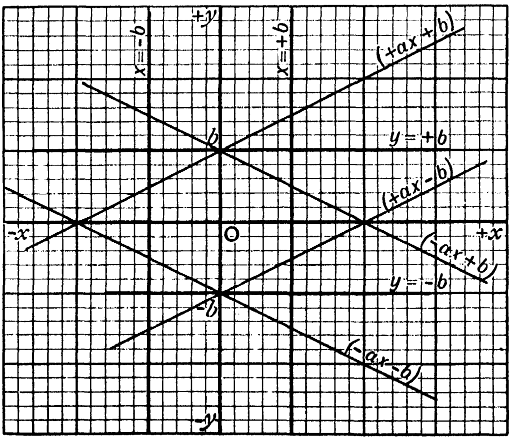


FIG. 5.

The various forms that the graph of $y = ax + b$ may take are shown in fig. 5.

Exercise X (a).

1. Without drawing the graphs, compare the gradients of the graphs of the following expressions, and state where each graph will cut the axes of y and x:

   (i) $3x-5$. (ii) $-x+4$. (iii) $\frac{2}{3}x-7$.

2. Write down the equations of graphs which have the following properties:

   (i) Gradient $+3$, intersects the axis of y at 7 below the origin.

   (ii) A down gradient of 3, intersects the axis of y at 5 above the origin.

   (iii) An up gradient of 5, intersects the axis of y at 3 below the origin.

   (iv) Gradient -2, intersects the axis of x at $+6$.

   (v) Gradient $2\frac{1}{2}$, passes through the origin.

3. A graph is parallel to the axis of x and intersects the axis of y at $+5$. What is its equation?

4. Write down the equation to each of the graphs in fig. 5.

5. Compare the graphs of the following expressions:

   $$3x-5, \quad 3x+2 \quad \text{and} \quad 2x-5.$$

6. What changes are made in a graph when the expression is doubled?

7. Plot the graph of the equation $x=2y+3$.

   What change occurs when the coefficient of y is reduced until it becomes 0?

8. Write down the equation to the graph obtained in Exercise IV (g), No. 2, and also to those in figs. 18 and 20, Chapter IV.

§ 5. Intersection of Graphs.

Since a point on the graph of an expression gives the value of the expression for that particular value of x, it follows that, at the point of intersection of two graphs, the expressions which they represent must have the same value.

Moreover, the x co-ordinate of the point of intersection must be the value of x for which the expressions have the same value, that is, are equal.

Draw the graphs of $2x+5$ and $-3x+25$ on the same axes, and verify this.

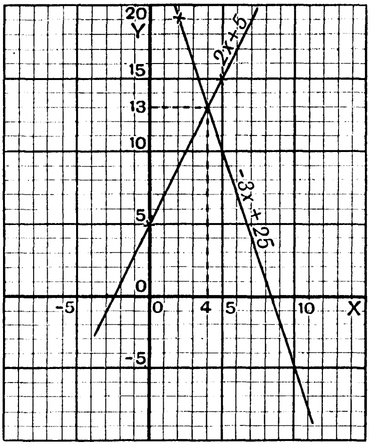


FIG. 6.

The value of x for which two expressions are equal may be found quickly by equating the expressions and solving the equation. Thus, taking the above expressions:

$$2x+5=-3x+25,$$

$$2x+3x=25-5,$$

$$x=4.$$

For this value of x, $\quad 2x+5=13,$

and $\quad -3x+25=13.$

§6. Interpolation and Extrapolation.

When the value of a function of, say, x is determined for a value of x between values for which the values of the function are already known, the process is called *Interpolation.*

On the other hand, when the value of x is not between values of x for which the values of the function are known, the process is called *Extrapolation.* Values determined by interpolation or by extrapolation are not necessarily correct.

For example, the graph given in fig. 7 shows the temperature of a quantity of water when heated by electrical means for the time shown. Readings were taken every 5 minutes for 20 minutes.

If the graph is produced, the temperature indicated by it corresponding to a period of 25 minutes is 116° C. But this is incorrect, for water boils at 100° C., and the temperature does not rise above this.

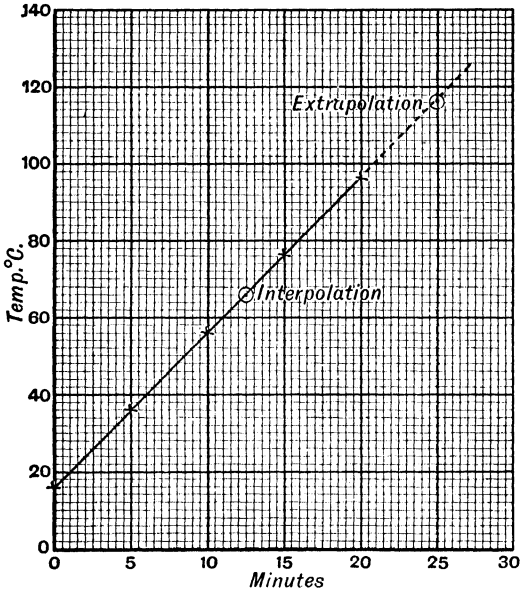


FIG. 7.

The correctness of a result obtained by extrapolation or interpolation depends upon whether the function is continuous throughout values which include those of the point under consideration.

Note.—In drawing a graph it is not necessary to adopt always the same scale on both axes or to number the point of intersection of the axes 0, 0.

EXERCISE X (B)

1. Why must the expressions of graphs which intersect have unequal coefficients of x?

2. Write down the expressions of two graphs which will not intersect.

3. State whether the graphs of $-2x+5$ and $2x-6$ will intersect, and if so, find the co-ordinates of the point of intersection.

Find graphically the values of x for which the following functions have the same value. Check by calculation.

4. $2x+3$ and $4x-3$.

5. $\frac{3}{2}x+2$ and $3x-1$.

6. $4-3x$ and $2(x+8)$.

7. $4x-18$ and $-(3x+10)$.

8. Draw a graph showing the amount of income tax on salaries from £160 to £500 at 9d. in the £, for excess over £160. Show graphically the effect of the abatement of £20 per annum per child, for three children.

9. Draw a graph showing the cost of articles to 1000 at 3d. each.

Show the effect of a reduction at the rate of 2/6 per 100 after the first 100.

10. Solve by a graphic method the following question:

A and B are approaching each other on the same road, A walking at 5 miles an hour and B cycling at 9 miles an hour. If B is at the second milestone when A is at the twentieth, to which milestone will they be nearest when they meet?

11. Solve by a graphic method the following problem:

A cyclist A is riding on a road out of a certain town at a rate of 8 miles per hour. A second cyclist B rides out on the same road at a uniform rate of 10 miles an hour. If B passes the first milestone ten minutes later than A, which will be the milestone nearest to them when B overtakes A?

12. One clock, A, gains and another, B, loses uniformly. At noon on Monday, A is 30 minutes slow and B 50 minutes fast. At noon on the following Friday, A is 10 minutes fast and B 10 minutes slow. Represent days on the axis of x and minutes fast and slow on the axis of y, and find graphically (i) the day and actual time at which the clocks indicated the same time, (ii) what that indicated time was.

Write down the equations to the graphs, and check your results by Algebra.

13. Plot the points $x=1$, $y=5$ and $x=-4$, $y=-5$. Join them by a straight line, and determine its equation.

14. Write down the equation to the straight line which cuts the axis of x at a distance 4 and the axis of y at a distance -6 from the origin.

15. Determine the equation to the straight line which passes through the points $x=1$, $y=1$ and $x=-3$, $y=9$.

16. What is the equation to the straight line at right angles to that of the last question, and which passes through the origin?

17. Calculate the distance between the points

$$x=2,\ y=5 \quad \text{and} \quad x=6,\ y=8.$$

18. Plot the point $x=-2y=6$ and the point $y=-\frac{3}{2}x=9$, and find the equation to the straight line joining them.

§ 7. Applications.

Being able to write down the equation to a given straight-line graph, you are now in a position to understand its use in Science and Mathematics.

EXAMPLE i.—The following numbers were obtained when a Fahrenheit and a Centigrade thermometer were used to determine, at various times, the temperature of a quantity of water which was being heated. The thermometers were read simultaneously.

Centigrade	15	20	25	30	40	60	80	100
Fahrenheit	59	68	77	86	104	140	176	212

Represent the Centigrade readings on the axis of x and the Fahrenheit on the axis of y, and plot points which have as co-ordinates these simultaneous values.

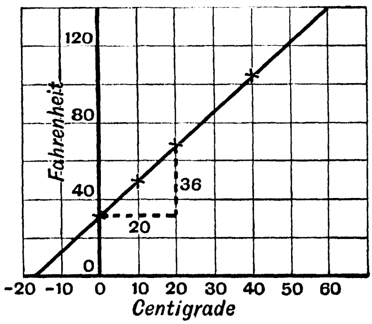


FIG. 8.

The points lie almost on a straight line. (Any deviation may be due to careless reading or imperfections of the thermometers)

Draw the straight line which passes evenly between the points, and find its equation.

The interpretation is that Centigrade readings are converted into Fahrenheit by multiplying by 1·8 or $\frac{9}{5}$ and adding 32.

EXAMPLE ii.—The following numbers show the volume of a mass of gas when heated to different temperatures, the pressure being constant:

Temperature(°C.)	15	25	35	40	60
Volume in c.cs.	150	155·25	160·5	163	173

Find the equation, and from it determine the temperature at which the volume of the gas would be zero.

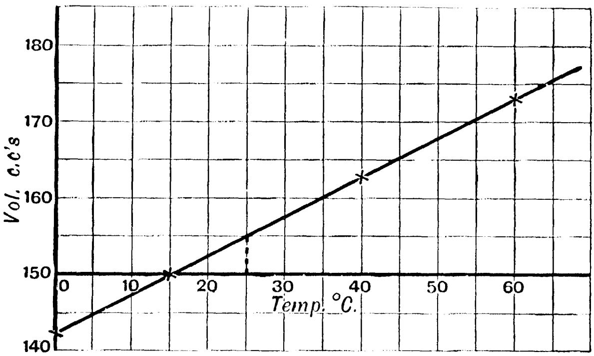


FIG. 9.

The equation found being $y = 0{\cdot}52x + 142$,

we have $0 = 0{\cdot}52x + 142$,

from which $x = -273{\cdot}3$,

i.e. 273·3 degrees below 0° C.

EXAMPLE iii.—The table gives the total heat in a pound of steam at different temperatures:

Temperature (°C.)	80	100	110	120	130	150
Total heat units	631	637	640	643	646	652

Find the law connecting total heat and temperature.

You will find it inconvenient to make the intersection of the axes 0 for either axis (fig. 10).

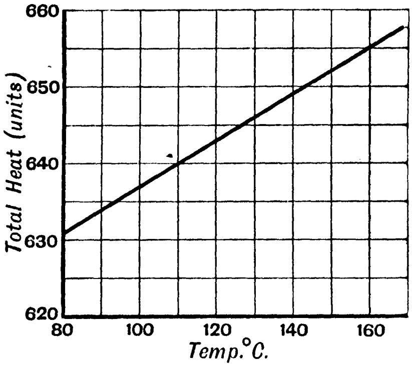


FIG. 10.

The added constant can be found as follows:

The gradient will be found to be 0·3 (approx.).

Let b represent the constant; then

$$y = 0{\cdot}3x + b.$$

Take two known values of x and y, and solve the equation for b. Thus:

$$637 = (0{\cdot}3 \times 100) + b,$$

from which $b = 607$.

Hence the relation is $y = 0{\cdot}3x + 607$.

The accepted relation is $y = 0{\cdot}305x + 606{\cdot}5$.

EXERCISE X (C)

1. The force to raise a roller up an inclined plane, the height of which was varied, was found to be as follows:

Height (cms.)	0	10	20	30	35
Force (grms.)	8	16	24	32	36

Find the law connecting height and force.

2. A body moves with a uniform velocity of 5 feet per second.

Draw a graph showing the distance covered in various intervals of time.

This graph is called the graph of positions.

Observe that the gradient is equal to the value of the velocity.

3. Using a set of pulleys, the force required to lift different weights was found to be as stated below:

Weight (lbs.)	0	8	12	20	52
Force (lbs.)	1·5	2·2	4·1	5·9	12·8

Find the equation connecting weight and force.

4. The following were the readings of a barometer when lowered into water:

Depth (ins.)	0	1	2	5	8	10	15	20	25	30
Reading (ins.)	30·0	30·07	30·15	30·37	30·59	30·7	31·07	31·46	31·83	32·2

Find the relation between the reading and the depth.

5. The following temperatures were taken every minute when a quantity of water was heated by a flame of constant power.

Find the equation connecting the temperature and the time of heating.

Time (mins.) -	0	1	2	3	4	5	6	7	8
Temperature (°C.)	15	18	21	24	27	30	33	36	39

6. The latent heat of steam at different temperatures is given in the following table. Find the law.

Temperature (° C).	100	120	140	160	180	200
Latent heat -	537	523	509	495·8	481	468

7. The table below gives the resistance of a length of platinum wire when its temperature is varied. Establish the equation connecting resistance and temperature.

Temp. (° C.)	15	20	40	60	80	100	120	150
Resistance (ohms)	105·2	106·9	113·8	120·7	127·6	134·5	141·3	151·7

8. In an experiment to determine the coefficient of expansion of benzene, the following numbers were obtained:

Temperature (° C.)	0	20	40	60	80
Volume - -	1·0	1·0241	1·0500	1·0776	1·1070

Plot these numbers, and find the equation connecting them.

§ 8. Graph of Inverse Proportion.

Plot the graph of the expression, $\frac{1}{x}$, i.e. of the equation:

$$y = \frac{1}{x}.$$

-3	-2	-1	$-\frac{1}{2}$	$=x=$	0	$\frac{1}{2}$	1	2	3	4	5
$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	$=\frac{1}{x}=$	$\frac{1}{0}$* $=\infty$	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

On plotting the points, you find that there are two distinct graphs; one in the first quadrant, and the other in the third (fig. 11).

Examine the graphs, and verify that they possess the following characteristics:

(i) Taking any two ordinates, and the corresponding abscissae, the ratio of the ordinates is equal to the inverse ratio of the corresponding abscissae. (This is why the graph is called the graph of Inverse Proportion.)

(ii) Each graph has a bend, or elbow, opposite the origin.

* Any number divided by 0 is equal to infinity; no finite number is large enough to represent the result. Infinity is denoted by the symbol ∞.

(iii) On each side of the elbows, the graphs get straighter and approach the axes, but never actually meet them. It is usual to say that the graphs cut the axes at infinity.

Each curve is called a **hyperbola**.

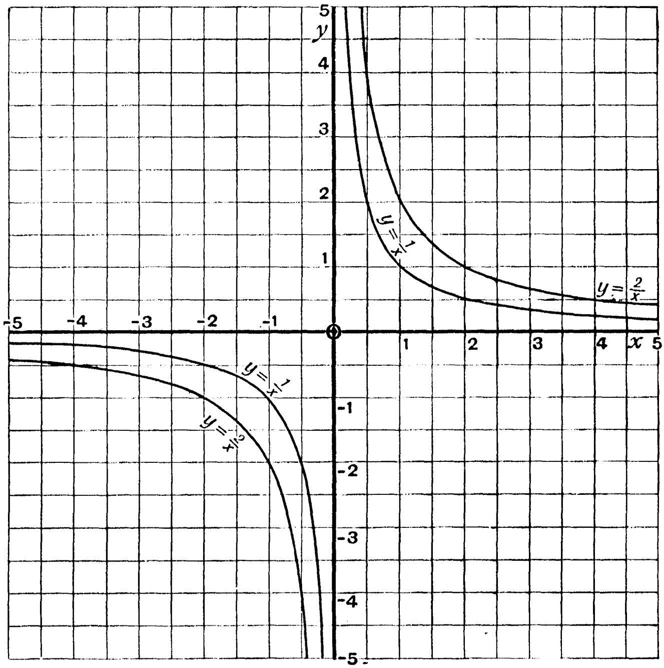


FIG. 11.

§ **9.** Plot now the graphs of the equation:

$$y=\frac{2}{x}.$$

Comparing these graphs with those of $\frac{1}{x}$, it is seen that the effect of the 2 has been to move the graphs away from the axes (fig. 11).

In the same way, you will find that the graphs of $\frac{1}{2x}$ are nearer the axes, but in neither case do the graphs cut the axes.

§**10.** To find the effect of an added constant, plot the graphs of :

$$\text{(i) } y=\frac{2}{x}+3. \quad \text{(ii) } y=\frac{2}{x}-3.$$

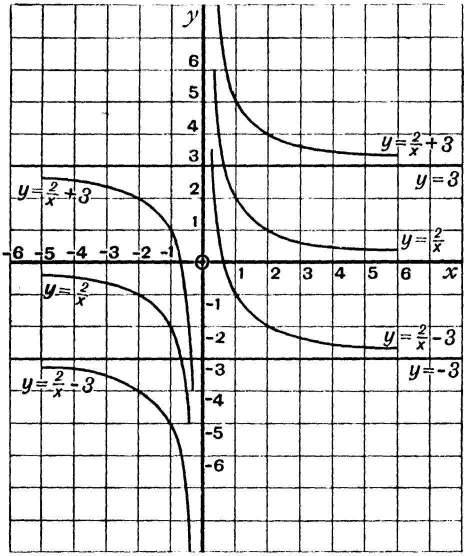


Fig. 12.

Comparing these with the graphs of $\frac{2}{x}$, it will be seen that the effect of the added constant has been :

(i) To raise the graphs, if the sign of the constant is +.

(ii) To lower the graphs, if the sign of the constant is −.

It will be noticed, also, that the graphs now approach the straight line $y=3$ in one case, and $y=-3$ in the other, instead of the axis of x. Observe that the graphs now cut the axis of x.

§**11.** If, in fig. 11, the names of the axes are interchanged, the graphs shown will then represent the equation $x=\frac{1}{y}$.

§ **12.** To find the effect of adding a constant to x, it is easier to examine it as follows:

If $$y = \frac{1}{x+3},$$

then $$x + 3 = \frac{1}{y},$$

and $$x = \frac{1}{y} - 3.$$

It will be concluded that the effect is similar to that examined in § 10, but with respect to the axis of y. If the axes are interchanged, the case is the same as the last examined.

§ **13. Applications.**

If, when simultaneous values of two quantities are plotted, the graph appears to be like that on page 109, the truth of the assumption that the law is $y = \frac{a}{x}$ * can be verified as follows:

If $y = \frac{a}{x}$, then, if z is written for $\frac{1}{x}$ (the reciprocal of x), $y = az$.

Hence, if y and z are plotted, the graph is a straight line of gradient a.

If the law is $y = \frac{a}{x} + b$, then, substituting as before, $y = az + b$, the graph of which is again a straight line of gradient a, but which cuts the axis of y at b.

Referring to the graph of $\frac{2}{x} + 3$ (page 110), plot y and $\frac{1}{x}$, and verify the above statement.

-4	-3	-2	-1	$=x=$	0	1	2	4
$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	$=\frac{1}{x}=$	∞	1	$\frac{1}{2}$	$\frac{1}{4}$
$2\frac{1}{2}$	$2\frac{1}{3}$	2	1	$=y=$	∞	5	4	$3\frac{1}{2}$

You find that the graphs for positive and negative values of x form a continuous straight-line graph which cuts the axis of y at 3, and has a gradient of 2 (fig. 13).

* Another way of expressing this relation is $xy = a$.

The equation is, therefore, $y = 2\left(\frac{1}{x}\right) + 3$, i.e. $y = \frac{2}{x} + 3$.

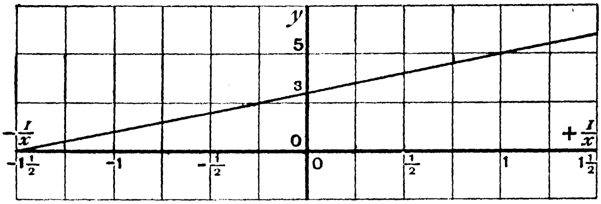


FIG. 13.

EXERCISE X (D)

Construct graphs to show the following:

1. The number of rails of different lengths required for a mile of railway.
2. The number of revolutions made by wheels of different diameters in covering a fixed distance.
3. The speed of a moving body and the time it takes to pass over a fixed distance.
4. The following numbers were obtained in an experiment for ascertaining how the volume of a gas changes when the pressure is varied:

P.	10	15	20	25	30	45
V.	45	30	22	18	15	10

Find the relation between pressure and volume.

5. The table shows the distance from a fulcrum at which a given weight must be placed to give a certain leverage.

Weight (lbs.) -	10	20	40	50
Distance (inches)	50	25	12·5	10

Find the relation between weight and distance.

6. A graph of the form $\frac{a}{x} + b$ passes through the points $x = 3$, $y = 3$ and $x = -1$, $y = -1$. Find its exact equation, and the straight lines which it approaches.

CHAPTER XI

SIMULTANEOUS SIMPLE EQUATIONS, LITERAL EQUATIONS, PROBLEMS

§1. Simultaneous Simple Equations.

We have seen on page 100 that it is possible for two otherwise different expressions to have equal values while the value of the unknown number is the same in both.

When the expressions form part of equations, the equations are called simultaneous equations.

Taking the expressions, $3x - 4$ and $2x + 5$; if we call the value of each expression y, we can write the example in equational form, thus:

$$y = 3x - 4,$$
$$y = 2x + 5;$$

or, transposing terms, thus:

$$3x - y = 4,$$
$$2x - y = -5;$$

or we might have different multiples of the equations, thus:

$$9x - 3y = 12,$$
$$-4x + 2y = 10,$$

where the first equation has been multiplied by 3 and the second by -2. When given such a pair of equations, the object is to find the values of x and y for which each equation holds good. In other words, to find the values of x and y which, when substituted for these symbols, make the expressions on the left equal to 12 and 10 respectively.

There are several methods of solving these problems.

METHOD I. This has been already indicated on page 101.

Find the value of either x or y in each equation, and equate the results.

$$9x - 3y = 12, \quad \text{.............................(i)}$$
$$-4x + 2y = 10. \quad \text{.............................(ii)}$$

From (i),

$$-3y = -9x + 12,$$
$$y = \frac{-9x + 12}{-3},$$
$$\text{i.e.} \quad y = 3x - 4. \quad \text{.......................(iii)}$$

From (ii), $$2y = 4x + 10,$$
$$y = 2x + 5. \quad \text{.......................(iv)}$$

Since each is equal to y,
$$\therefore\ 3x - 4 = 2x + 5,$$
$$x = 9.$$

y may now be found from either equation (iii) or (iv).

Thus, $$y = 2x + 5$$
$$= 2 \times 9 + 5$$
$$= 23.$$

Check this result by substituting these values in equations (i) and (ii).

METHOD II. From one equation obtain the value of one of the unknowns, say y, in terms of the other, and substitute this value in the other equation, thus obtaining an equation with only one unknown.

From (i), $$y = 3x - 4. \quad \text{.........................(iii)}$$

Substituting in (ii), $$-4x + 2(3x - 4) = 10,$$
$$-4x + 6x - 8 = 10,$$
$$2x = 18,$$
$$x = 9.$$

From (iii), $$y = 3 \times 9 - 4 = 23.$$

METHOD III. In this method the equations are multiplied by such numbers as will make the coefficients of one of the unknowns numerically the same. Then, if the signs of these coefficients are alike, by subtracting, or, if unlike, by adding, this unknown disappears, and an equation is obtained which contains one unknown only.

$$9x - 3y = 12, \quad \text{...........................(i)}$$
$$-4x + 2y = 10. \quad \text{.........................(ii)}$$

To get rid of y, multiply equation (i) by 2 (the coefficient of y in equation (ii)), and equation (ii) by 3 (the numerical coefficient of y in (i)). Then we have

$$18x - 6y = 24 \quad \text{.........................(iii)}$$
$$-12x + 6y = 30. \quad \text{.........................(iv)}$$

Adding, $$\therefore\ 6x = 54;$$
$$\therefore\ x = 9.$$

y is obtained from either equation (i) or (ii).

Further Examples.

EXAMPLE i.
$$\frac{x}{3} + 4y = -32,$$
$$6x + 5y = 27.$$

In order to clear the first equation of fractions, multiply both sides of it by 3.

Then
$$x + 12y = -96,$$
$$6x + 5y = 27.$$

These can now be solved by one of the methods given.

EXAMPLE ii.
$$\frac{2}{x} + \frac{5}{y} = 7, \quad \text{.....................(i)}$$
$$\frac{3}{x} - \frac{2}{y} = 11. \quad \text{.....................(ii)}$$

In such a case it is better to find the value of $\frac{1}{x}$.

Multiply equation (i) by 3 and equation (ii) by 2.

Then
$$\frac{6}{x} + \frac{15}{y} = 21$$
$$\frac{6}{x} - \frac{4}{y} = 22.$$

Subtracting,
$$\therefore \frac{19}{y} = -1;$$
$$\therefore \frac{1}{y} = \frac{-1}{19},$$

and
$$y = -19.$$

Substituting in (i),
$$\frac{2}{x} - \frac{5}{19} = 7,$$
$$\frac{2}{x} = 7 + \frac{5}{19},$$
$$\frac{2}{x} = \frac{138}{19},$$
$$\frac{1}{x} = \frac{138}{38};$$
$$\therefore x = \frac{38}{138} = \frac{19}{69}.$$

Verify by substituting these values in equations (i) and (ii).

A modification of this method is that of writing a for $\frac{1}{x}$ and b for $\frac{1}{y}$, and then finding a and b, from which x and y are quickly found by inverting the values found.

Example iii.—The given lines s and d represent respectively

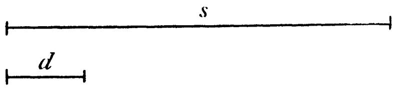


Fig. 1.

the sum and difference of two other lines. Find the unknown lines.

An algebraic consideration will show us the method.

Let the length of the longer unknown line be x units,
and that of the shorter " " y units.

Then $$x + y = s \quad \text{.......(i)}$$
and $$x - y = d. \quad \text{.......(ii)}$$

Adding, $$2x = s + d;$$

$$\therefore \ x = \frac{s + d}{2}.$$

That is, the longer line is half the sum of s and d.

To find this line geometrically, add d to s, and bisect the whole line. y is readily found from equation (i).

Example iv.—When there are three unknowns, three different equations are required. From these, unknowns can be eliminated until only one remains. Thus:

$$3a - 2b + c = 1, \quad \text{.......(i)}$$
$$2a - 3b - c = -6, \quad \text{.......(ii)}$$
$$a + 5b + 3c = 20. \quad \text{.......(iii)}$$

Add (i) and (ii), and c is eliminated.

$$\begin{array}{rrrl} 3a - & 2b + & c = & 1 \\ 2a - & 3b - & c = & -6 \\ \hline 5a - & 5b & = & -5 \quad \text{......(iv)} \end{array}$$

Multiply (i) by 3 and subtract (iii).

$$\begin{array}{rrrl} 9a - & 6b + & 3c = & 3 \\ a + & 5b + & 3c = & 20 \\ \hline 8a - & 11b & = & -17 \quad \text{......(v)} \end{array}$$

Equations (iv) and (v) are now readily solved.

To find c substitute the values of a and b in one of the original equations.

The process is the same for any number of unknowns.

Exercise XI (a)

Solve the following simultaneous equations:

1. $3x - 2y = 18,$
$2x - 3y = -1.$

2. $3a + 5b = 19,$
$5a - 4b = 7.$

3. $\frac{x}{3} = \frac{y}{2},$
$\frac{x}{6} + 2y = 9.$

4. $\frac{a}{5} + 5b = -4,$
$5a + \frac{b}{5} = 4.$

5. $2x - 5y + 4 = 0,$
$3x + 2y = 7.$

6. $3x - 2y = 1,$
$4x + 3y = 41.$

7. $x + \frac{1}{y} = \frac{1}{5},$
$5x + \frac{2}{y} = \frac{7}{10}.$

8. $\frac{1}{5}(2x + 3y) = \frac{1}{6}(x + 3y + 3) = \frac{1}{2}(9y - x + 1).$

9. $\frac{2x+1}{7} = \frac{3y-2}{4},$
$\frac{x+2y-1}{3} = \frac{3x-y+3}{5}.$

10. $a + 2b - 3c = 6,$
$2a + 4b - 7c = 9,$
$3a - b - 5c = 8.$

11. $3(x+y) + 5(x-y) = 19,$
$5(x+y) - 4(x-y) = 7.$

12. $\frac{a+b}{4} + \frac{a-b}{3} = 10,$
$\frac{a+b}{8} - \frac{a-b}{6} = 5.$

13. $\frac{2}{3}x - \frac{4}{5}y = 2,$
$\frac{5}{6}x - \frac{7}{8}y = 4.$

14. $x + 2y = 0,$
$\frac{8}{x} + \frac{5}{y} = 1.$

15. $7a - 3b = 30,$
$9b - 5c = 34,$
$\frac{a}{3} + \frac{b}{3} + \frac{c}{3} = 11.$

16. $\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1,$
$\frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 24,$
$\frac{8}{x} - \frac{6}{y} + \frac{6}{z} = 15.$

17. $2x + 4y - 3z = -7,$
$5x + 3y - 2z = 10,$
$7x + 4y - 5z = 3.$

18. $\dfrac{x}{3} + \dfrac{y}{4} = \dfrac{3}{2},$
$2x - y = 19.$

§ 2. Literal Equations.

In Literal Equations, letters other than those which represent the unknowns are introduced. Thus:

EXAMPLE i.—$a(x - b) - b(x - a) = a^2 - b^2$. Find x.

$$ax - ab - bx + ab = a^2 - b^2,$$
$$x(a - b) = a^2 - b^2,$$
$$x = \frac{a^2 - b^2}{a - b} = a + b.$$

In this and similar examples, all the letters, except x, are treated like the numbers in previous examples on equations.

Arrange on one side only, all terms containing x. Carry all other terms to the other side. Bracket the terms containing x, and take x outside the bracket. The rest is easy. Verify the above result.

EXAMPLE ii.
$$ax + by = 2ab, \quad \text{(i)}$$
$$-bx + ay = a^2 - b^2. \quad \text{(ii)}$$

To eliminate y, multiply equation (i) by a, and equation (ii) by b, and subtract.

$$a^2x + aby = 2a^2b$$
$$-b^2x + aby = a^2b - b^3$$
$$a^2x + b^2x = a^2b + b^3,$$
$$\therefore \ x(a^2 + b^2) = b(a^2 + b^2);$$
$$\therefore \ x = b.$$

From (i),
$$ab + by = 2ab,$$
$$by = ab;$$
$$\therefore \ y = a.$$

Verify this result by substituting these values in equation (ii).

EXERCISE XI (B)

1. $a(x - a) - b(x - b) = a - b$. Find x.

2. $\dfrac{3a^2b}{5cx} = \dfrac{1}{ab}$. Find x.

3. $b\dfrac{b - x}{c} - c\dfrac{c + x}{b} = x$. Find x.

4. $\frac{x}{2a}+\frac{y}{2b}=1$ and $bx=ay$. Find x and y.

5. $\frac{x}{a}+\frac{y}{b}=\frac{x}{b}+\frac{y}{a}$. Find $\frac{x}{y}$.

6. $x-y=a-b$ and $bx+ay=2ab$. Find x and y.

7. If $2s=a+b+c$, show that:

(i) $a+b-c=2(s-c)$.

(ii) $b+c-a=2(s-a)$.

(iii) $c+a-b=2(s-b)$.

8. If $c=2\pi r$, find r.

9. If $A=\frac{\pi d^2}{4}$, find d.

10. If $c=\pi(a+b)$, find a.

11. If $A=\pi ab$, find b.

12. If $A=4\pi r^2$, find r.

13. If $V=\frac{4}{3}\pi r^3$, find r.

14. $Q=Ws(t_1-t_2)$.*

Arrange this for calculating respectively:

(i) s. (ii) t_1. (iii) t_2.

15. $Ws(T-x)=w(x-t)$.

Arrange this in convenient forms for calculating:

(i) x. (ii) w. (iii) t. (iv) s.

16. $WL+W(T-x)=w(x-t)$.

Arrange this for the determination of: (i) L. (ii) x.

17. $WL+W(T-x)=w_1(x-t)+w_2s(x-t)$.

Arrange this for the calculation of: (i) L. (ii) s. (iii) x.

18. $L=l(1+at)$. Find a.

19. $V=v(1+bt)$. If $V=\frac{W}{D}$ and $v=\frac{W}{d}$, substitute these values for V and v, and find the equation connecting D and d.

20. $H=k\frac{A(T-t)}{l}$.

Arrange this equation for finding: (i) k. (ii) T.

21. $s=ut+\frac{1}{2}at^2$.

Arrange this equation for calculating: (i) a. (ii) u.

What does the equation become when $u=0$ and $a=32$?

* t_1 and t_2 represent two different values of t. The figures 1 and 2 are neither coefficients nor indices.

22. $v^2 - u^2 = 2as$.

Arrange this for determining: (i) s. (ii) a. (iii) v.

23. What does the equation $F = ma$ become when $a = \frac{v-u}{t}$?

Arrange the equation for finding v.

24. $\frac{1}{2}W(v^2 - u^2) = Fs$.

Arrange this for calculating F.

What does the equation become when $F = Wg$?

25. $\frac{W}{v} = d$ and $\frac{W}{V} = D$, and $V = v(1 + ct)$.

Show that $D = \frac{d}{1+ct}$.

26. $F = \frac{m}{r^2} - \frac{m}{R^2}$. Find m.

27. $\frac{1}{u} + \frac{1}{f} = \frac{1}{v}$.

Arrange this for finding: (i) f. (ii) v.

28. If $\frac{1}{u} - \frac{1}{f} = \pm\frac{1}{v}$, when will v be + and when − ?

29. If $3x - 2 = A(x-4) + B(x+1)$ for all values of x, find A and B.

Hint: For finding B, *take* $x = 4$.

30. $y = a + bx$. If, when $y = 113$, x is 230, and when y is 206, x is 320, find a and b.

What is the value of y when $x = 212$?

31. $\text{Density} = \frac{\text{Weight}}{\text{Volume}}$.

Find the density of the alloy formed by fusing x grams of lead of density l with y grams of tin of density t.

§ 3. Problems.

Problems, which in many cases appear very difficult when an attempt is made to solve them by the rules of Arithmetic, often yield readily to algebraic treatment.

Consider the following simple example:

When two more passengers enter a railway compartment, it contains three times as many persons as it would have done if, instead, four had alighted. How many were there originally in the compartment?

All such problems contain sufficient information to enable you to represent symbolically, all the unknown numbers mentioned, and to form equations from which they can be calculated.

Proceed as follows:

Represent symbolically, all the unknown numbers.

It is wise to introduce as few symbols as possible.

Let $x =$ the number of passengers originally present.

Then $(x+2) =$ the number after 2 more have entered,
and $(x-4) =$ the number if, instead, 4 had alighted.

Refer to the problem for the relation which exists between these numbers.

The problem states that $(x+2)$ is three times $(x-4)$,

$$\begin{aligned} \text{i.e.}\quad (x+2) &= 3(x-4), \\ x+2 &= 3x-12, \\ -2x &= -14, \\ x &= 7. \end{aligned}$$

The result can be verified by testing whether it satisfies the conditions of the problem.

Thus, the problem states that when there are 2 more, i.e. 9 passengers, there are three times as many as there would have been had 4 alighted, i.e. three times 3.

The result satisfies the conditions of the problem.

Statements such as: "One number is so many times another," "One number exceeds another by so much," "The result is the same as," etc., suggest equality, and therefore an equation.

It is important to read the problem carefully, and to write the numbers in symbolic form before attempting to form an equation.

Matters of Importance.

1. State the units when possible.

E.g. Let $x =$ the number of shillings, grams, minutes, etc.

There is no objection to writing £x.

2. The same digit may, in one case, represent units, in another, tens, and so on.

Thus, if the digits of a number be x and y, the number may be $10x+y$, or $10y+x$.

3. As many equations are required as there are symbols.

4. Odd and even numbers.

If n represents any number, odd or even, $2n$ will be even, for twice an odd and twice an even number are both even.

It follows that $(2n+1)$ will always be odd, for the number obtained by adding one to an even number is always odd.

When there are two or more unknowns and the relation between them is not simple enough to allow you to represent them in terms of the symbol chosen for one, it is better to represent them by different symbols. As many equations as there are unknowns are then necessary to determine the unknowns.

EXAMPLE.—*A whole number consists of two digits, and is such that the sum of its digits is one less than one-third of the number, and if* 15 *is added to twice the number, the digits are reversed. Find the number.*

Let $x=$ the first digit and $y=$ the second digit.

Then, $10x+y=$ the required number.

From the first statement,

$$x+y+1=\frac{10x+y}{3} \quad \text{.......................(i)}$$

From the second statement,

$$2(10x+y)+15=10y+x. \quad \text{.....................(ii)}$$

The equations, when simplified, give

$$7x-2y=3,$$
$$19x-8y=-15,$$

from which it will be found that

$$x=3 \quad \text{and} \quad y=9.$$

The number is therefore 39.

EXERCISE XI (c)

1. If a cyclist cover x miles in y hours, how many yards does he cover per minute?
2. If one train travels at the rate of x miles per hour and another at y yards per minute, what is the difference in their speed in feet per second?
3. If one metre measures 39·37 inches, find the difference in yards between x miles and x kilometres.
4. A gallon of water weighs 10 lbs., and a cubic foot, 62·5 lbs. Find in gallons the difference between x cubic feet and x gallons.
5. When a certain number is increased by 8, the result is the same as when its double is diminished by 1. Find the number.

6. A straight line, 1 foot long, is divided into two parts such that the difference between the parts is an inch longer than one quarter of the smaller. Find the parts.

7. A certain number consists of two digits, and when 18 is added the digits are reversed. What is the difference between the digits? If the second digit is twice the first, what is the number?

8. If a train had travelled 15 miles an hour faster, it would have journeyed half as far again as it did. Find the speed, and the distance actually covered in 6 hours.

9. The speed of a certain wheel, when running down, is found to decrease proportionally with time. At a certain instant its speed is taken, and again two minutes afterwards, when it is found to have decreased by a quarter. If, in coming to rest, it makes 1200 revolutions from the time the speed was first taken, find the original speed.

10. The sum of four consecutive odd numbers is 48. Find them.

11. Referring to the worked example on page 122, the digits can be found if it is borne in mind that they are whole numbers. Find them.

12. Find two numbers such that one-third of the first, increased by 6, is equal to one half the second, diminished by 3, and such that their sum is 2 less than five times their difference.

13. When 50 c.c. of lead and 10 c.c. of tin are fused together, the total weight is 643 grams. When 80 c.c. of lead and 20 c.c. of tin are fused together, the total weight of the alloy is 1058 grams. Find the density of lead and of tin.

14. If a certain rectangular plot of ground were 4 yards longer and 2 yards wider, it would contain 108 more square yards.

If it were 6 yards longer and 6 yards wider, it would contain 246 more square yards. Find its dimensions.

15. A man buys a dozen eggs, some of which he finds to be bad. Had he received only the good eggs for his money, the price per dozen would have been a third as much again. Find the number of bad eggs.

16. By selling a bicycle, a man gained 5 per cent. What would he have gained per cent. had he sold it for half as much again?

CHAPTER XII

FACTORS, FRACTIONS

§ 1. Factors.

It is already known that:

$$(a+b)(c+d) = a(c+d) + b(c+d)$$
$$= ac + ad + bc + bd.$$

Hence the factors of $ac+ad+bc+bd$ are $(a+b)$ and $(c+d)$.

Now commence with the expression $ac+ad+bc+bd$, and retrace the steps.

i. Bracket in pairs.	$(ac+ad)+(bc+bd)$
ii. Take the common term a out of the first bracket and b out of the second bracket.	$=a(c+d)+b(c+d)$
iii. $c+d$ is common to both terms.	$=(a+b)(c+d)$.

The factors are $(a+b)$ and $(c+d)$.

Note.—The bracketed expressions in the second line must be exactly alike.

EXAMPLE.—Find the factors of $2ax - ay - 4bx + 2by + 2x - y$.

$$2ax - ay - 4bx + 2by + 2x - y$$
$$= (2ax - ay) - (4bx - 2by) + (2x - y)$$
$$= a(2x - y) - 2b(2x - y) + 1(2x - y)$$
$$= (a - 2b + 1)(2x - y).$$

The factors are $(a - 2b + 1)$ and $(2x - y)$.

EXERCISE XII (A)

Find the factors of:

1. $x^2 - x$. 2. $a - ax$. 3. $a^2 + ax$. 4. $a^2x - ax^2$. 5. $x^3 + x$.

6. $2a^2 - 4ab + 3ac - 6bc$. 7. $ax - ay - 3x + 3y$.

8. $a^4 - a^3 + 3a - 3$. 9. $2x^2y + 2ay - 5x^2 - 5a$.

10. $x^2 - ax - 3a + 3x$. 11. $a^4 + a^3b - 2ab^3 - 2b^4$.

12. $2x^4 + 24x - 8x^2 - 6x^3$. 13. $a^4x + 4bcx - b^2x - 4c^2x$.

14. (i) $a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3$.

(ii) $a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3$.

What, therefore, are the factors of $a^3 - b^3$, and of $a^3 + b^3$?

Factors of a trinomial expression.

EXAMPLE i. $x^2 + 3x + 2$.

If one of these terms can be split into two, then it might be possible to employ the method of grouping.

The middle term $3x$ may be written as $2x + x$; then

$$\begin{aligned} x^2 + 3x + 2 &= x^2 + 2x + x + 2 \\ &= x(x+2) + 1(x+2) \\ &= (x+1)(x+2). \end{aligned}$$

The difficulty will be in finding how to split up the middle term.

The following rule is sound:

Multiply the first and last terms of the trinomial; split this product into two factors, such that the sum is equal to the middle term.

Thus (i) $x^2 \times 2 = 2x^2$. (ii) $2x^2 = 2x \times x$, and $2x + x = 3x$.

EXAMPLE ii. $x^2 - 4x - 12$.

$$\begin{aligned} \text{First} \times \text{last} &= -12x^2, \\ \text{factors} &= -6x \text{ and } +2x, \\ \text{sum} &= -4x. \end{aligned}$$

Then,
$$\begin{aligned} x^2 - 4x - 12 &= x^2 - 6x + 2x - 12 \\ &= x(x-6) + 2(x-6) \\ &= (x+2)(x-6). \end{aligned}$$

EXAMPLE iii. $9x^2 + 3x - 2$.

$$\begin{aligned} \text{First} \times \text{last} &= -18x^2, \\ \text{factors} &= 6x \text{ and } -3x, \\ \text{sum} &= 3x. \end{aligned}$$

$$\begin{aligned} 9x^2 + 3x - 2 &= 9x^2 + 6x - 3x - 2 \\ &= 3x(3x+2) - 1(3x+2) \\ &= (3x-1)(3x+2). \end{aligned}$$

EXAMPLE iv.—Special case: $a^2 - b^2$.

Here the middle term is missing, i.e. it is 0.

The expression may be written $a^2 \pm 0 - b^2$, or $a^2 \pm 0ab - b^2$.

$$\begin{aligned} \text{First} \times \text{last} &= -a^2b^2, \\ \text{factors} &= ab \text{ and } -ab, \\ \text{sum} &= 0. \end{aligned}$$

$$\begin{aligned} a^2 - b^2 &= a^2 - ab + ab \times -b^2 \\ &= a(a-b) + b(a-b) \\ &= (a+b)(a-b). \end{aligned}$$

This last case is so important that it is well to remember it in the following form:

The difference between the squares of two terms is equal to the product of their sum and difference.

Fig. 1 illustrates this result.

The figures ABCD and DEFG are squares of sides x and y units respectively. The shaded area ABCGFE represents

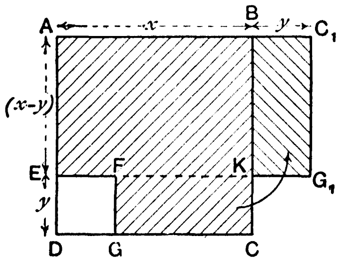


FIG. 1.

$(x^2 - y^2)$, and if this figure be cut along the dotted line FK, which is EF produced, and the part FKCG placed in the position BC_1G_1K, then the rectangle AC_1G_1E is obtained, the length of which, AC_1, is $(x+y)$ units, and the breadth, AE, $(x-y)$ units, and therefore the area $(x+y)(x-y)$ units.

EXERCISE XII (B)

Factorise:

1. $x^2 - 5x + 6$. 2. $x^2 + 5x + 6$. 3. $x^2 + x - 6$ 4. $x^2 + 7x + 6$.
5. $x^2 - x - 6$. 6. $x^2 - 7x + 6$. 7. $x^2 + 5x - 6$. 8. $x^2 - 5x - 6$.
9. $a^2 + 13a + 12$. 10. $a^2 - 13a + 12$. 11. $a^2 + 11a - 12$.

12. $a^2-11a-12$.
13. $a^2+8a+12$.
14. $a^2-8a+12$.
15. $a^2+4a-12$.
16. $a^2-4a-12$.
17. $a^2+7a+12$.
18. $a^2-7a+12$.
19. a^2-a-12.
20. a^2+a-12.
21. x^2-y^2.
22. x^2-4y^2.
23. $4x^2-y^2$.
24. x^2-1.
25. x^2y^2-1.
26. $4a^2-9b^2$.
27. $x^2-(y-1)^2$.
28. x^2-y^2-2y-1.
29. $9a^2x^2-4b^2y^2$.
30. $(a+x)^2-(a-x)^2$.
31. $(a-x)^2-(b+y)^2$.
32. $a^2+2ax+x^2-b^2+2by-y^2$.
33. $(x^2+6x+9-y^2+4y-4)$.
34. x^2-4x-y^2-6y-5.
35. $4x^2+12x+5-9y^2+12y$.
36. $x^4+x^2y^2+y^4$. Observe that

$$x^4+x^2y^2+y^4=x^4+2x^2y^2+y^4-x^2y^2.$$

37. $x^4-3x^2y^2+y^4$.
38. $x^2-2xy-8y^2$.
39. $6x^2-5xy-6y^2$.
40. $6x^2+9xy-6y^2$.
41. $12(x^2-y^2)-7xy$.
42. $(b-c)^3+9(c-b)$.
43. $4x^2+6xy-4y^2$.
44. $a^2-b^2+ac-bc$.
45. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$.
46. $(a+x)^4-(a-2x)^4$.
47. Show that:

$$4a^2b^2-(a^2+b^2+c^2)^2=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}$$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

If $a+b+c=2s$, show that the given expression equals

$$16s(s-a)(s-b)(s-c).$$

§ 2. An Important Matter concerning Factors.

Consider the example:

$$x^2-4x-12=(x-6)(x+2).$$

If we substitute 6 for x, then $x-6$, and therefore $(x-6)(x+2)$ becomes 0.

It follows that the expression $x^2-4x-12$ should have a value 0 for x equal to 6.

Verify this statement.

Similarly, the expression $x^2-4x-12$ is equal to 0 when x is equal to -2.

The converse is true, namely: If the value of an expression

containing x becomes zero when a value, say a, is substituted for x, then $x-a$ is a factor of the expression.

This gives you another method of testing the accuracy of factors.

EXAMPLES.

i. To show that $a-b$ is a factor of a^3-b^3.

Substitute b for a; then a^3-b^3 becomes b^3-b^3, which is equal to 0.

ii. To show that $a-b$ is a factor of

$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

Substituting b for a, the expression becomes

$$b^2(b-c)+b^2(c-b)+c^2(b-b),$$

which is seen to equal 0.

Similarly, show that $(b-c)$ and $(c-a)$ are factors.

§ 3. Cyclic Order.

In some expressions, the symbols recur in an order called cyclic.

Fig. 2 shows the symbols a, b, c, spaced round a closed curve.

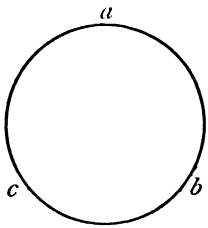


FIG. 2.

The symbols of such expressions as those given below follow round the curve in the same direction, namely, clockwise.

i. $(a-b)+(b-c)+(c-a)$. ii. $a(b-c)+b(c-a)+c(a-b)$.

iii. $ab(a-b)+bc(b-c)+ca(c-a)$.

The sum of the terms of such expressions is often written in the form, $\sum\limits_{abc} a(b-c)$, in which Σ (Greek letter sigma) means the sum, the terms being of the type indicated by the term $a(b-c)$, but completed in cyclic order for the three symbols a, b, c.

Thus, $\sum\limits_{abc} a(b-c)$ is a way of writing briefly Example ii. given above.

Exercise XII (c)

1. Show that $2x - 1$ is a factor of $6x^4 + x^3 - 8x^2 + 23x - 10$.
2. Show that $6x^4 + x^3 - 8x^2 + 23x - 10$ is exactly divisible by $x + 2$.
3. Find the factors of $x^4 + 4x^3 - 7x^2 - 22x + 24$.
4. Find the factors of $a^4 - 8a^3 + 17a^2 + 2a - 24$.
5. Show that $(a + b + c)$ is a factor of $a^3 + b^3 + c^3 - 3abc$.
6. Show that $(a - b)$, $(b - c)$, $(c - a)$ are factors of
$$a^3(b - c) + b^3(c - a) + c^3(a - b).$$
Are there other algebraic factors? Why?
7. Write in full the expressions:
$$\sum_{abc} ab(a - b), \quad \sum_{abc} a^2(b - c), \quad \sum_{abcd} (a - b), \quad \sum_{abcd} ab(b - c), \quad \sum_{abc} \frac{a}{b - c}.$$
8. Show that $\sum_{abc} (a - b) = 0$.
9. Show that (i) $\sum_{abc} a^2(b - c) = -(a - b)(b - c)(c - a)$.
(ii) $\sum_{abc} ab(a - b) = -(a - b)(b - c)(c - a)$.

§ 4. Application of Factors.

Solve the equation $x^2 - 3x + 2 = 0$, i.e. find the values of x for which $x^2 - 3x + 2$ is 0.

By factors, $(x - 2)(x - 1) = 0$.

For a product to give 0 as the result, at least one of the factors must be 0.

Hence $(x - 2)(x - 1)$ equals 0,

(i) when $(x - 2) = 0$, from which $x = 2$;
(ii) ,, $(x - 1) = 0$, ,, $x = 1$.

Check these results by substituting these values of x in turn in $x^2 - 3x + 2$.

Exercise XII (d)

Solve the following equations:

1. $x^2 - 5x + 6 = 0$.
2. $x^2 + 5x = -6$.
3. $2x^2 - 7x + 6 = 0$.
4. $x^2 - x = 12$.
5. $x^2 = 12 + 4x$.
6. $14x = 15 - x^2$.

7. $\frac{x^2}{4} - 3x + 8 = 0.$ 8. $3x^2 + 14x = 5.$

9. $x^3 - 2x^2 - 9x + 18 = 0.$ 10. $x^2 - 4 = 0.$

11. $x^2 + 4x - 10 = 11.$ 12. $x^3 + 8 = 0.$

13. $(x-3)(6x^2 + 5x - 6) = 0.$ 14. $x^2 - 3ax + 2a^2 = 0.$

15. $x^2 + ax = 2a^2.$ 16. $a^2x^2 + 3ax + 2 = 0.$

17. $(x+1)^2 = a^2.$ 18. $12x^2 - 17x + 8 = 2.$

19. $x^2 - (a+b)x + ab = 0.$ 20. $x^4 - 2x^3 + 2x^2 - 3x - 2 = 0.$

Examine the expression,

$$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5. \quad \text{................(i)}$$

It will be observed that, regarded from left to right, the powers of x decrease, and the powers of y increase. This arrangement is very convenient.

If the coefficient of any term is 0, the term vanishes.

If the coefficients of all terms between the end terms are 0, the expression given becomes $x^5 + y^5$. ....(ii)

Expression (ii) can be obtained from (i) by adding to (i), terms like the intermediate terms, but of opposite sign, thus:

$$\begin{aligned} x^5 + y^5 = x^5 + x^4y - x^4y - x^3y^2 + x^3y^2 + x^2y^3 \\ - x^2y^3 - xy^4 + xy^4 + y^5. \end{aligned}$$

1. It is readily seen that the right-hand side, and therefore $x^5 + y^5$, is divisible by $x + y$.

$$\begin{array}{r|l} & x^4 \qquad - x^3y \qquad + x^2y^2 \qquad - xy^3 \qquad + y^4 \ (\textit{Quot.}) \\ \hline x + y & x^5 + x^4y - x^4y - x^3y^2 + x^3y^2 + x^2y^3 - x^2y^3 - xy^4 + xy^4 + y^5 \\ & \underline{x^5 + 4x^4y} - \underline{x^4y - x^3y^2} + \underline{x^3y^2 + x^2y^3} - \underline{x^2y^3 - xy^4} + \underline{xy^4 + y^5} \\ & \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \end{array}$$

From the appearance of this example, it is evident that:

(i) Any similar expression having its terms to the same odd power is exactly divisible by $x + y$.

E.g. $x^9 + y^9$ is exactly divisible by $x + y$.

Verify this.

(ii) $x^5 - y^5$ is not exactly divisible by $x + y$. What is the remainder in this case? Similarly, any expression of the form $x^{\text{odd power}} - y^{\text{same odd power}}$, is not exactly divisible by $x + y$.

Examine the above quotient, and observe that:

(i) The terms are conveniently arranged in descending and ascending powers, and that no terms are missing.

(ii) The signs are alternately plus and minus.

The quotient of $\dfrac{x^7+y^7}{x+y}$ can be written down at once as:

$$x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6.$$

Similarly, $$\frac{x^3+y^3}{x+y} = x^2 - xy + y^2.$$

The factors of $x^3 + y^3$ are therefore $(x+y)$ and $(x^2 - xy + y^2)$.

2. In the same manner, it is shown that:

$x^5 - y^5$ is exactly divisible by $x - y$.

$$x^5 - y^5 = x^5 - x^4y + x^4y - x^3y^2 + x^2y^3 - xy^4 + xy^4 - y^5.$$

$$\begin{array}{r|l}
 & x^4 \qquad + x^3y \qquad + x^2y^2 \qquad + xy^3 \qquad + y^4 \\
\hline
x - y & x^5 - x^4y + x^4y - x^3y^2 + x^3y^2 - x^2y^3 + x^2y^3 - xy^4 + xy^4 - y^5 \\
 & \underline{x^5 - x^4y} + \underline{x^4y - x^3y^2} + \underline{x^3y^2 - x^2y^3} + \underline{x^2y^3 - xy^4} + \underline{xy^4 - y^5} \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

It follows $x^{\text{odd}} - y^{\overset{\text{same}}{\text{odd}}}$ is exactly divisible by $x - y$.

Question: Referring to the above example, is $x^5 + y^5$ exactly divisible by $x - y$? If not, what is the remainder?

Examine the quotient, and observe that:

(i) The terms run in natural order.

(ii) The signs are all plus.

The quotient of $\dfrac{x^3-y^3}{x-y}$ can be written down at once, thus:

$$\frac{x^3-y^3}{x-y} = x^2 + xy + y^2.$$

The factors of $x^3 - y^3$ are therefore $(x-y)$ and $(x^2 + xy + y^2)$.

3. $x^6 - y^6$ is exactly divisible by $x + y$.

$$x^6 - y^6 = x^6 + x^5y - x^5y - x^4y^2 + x^4y^2 + x^3y^3 - x^3y^3 - x^2y^4 + x^2y^4 + xy^5 - xy^5 - y^6.$$

$$\begin{array}{r|l}
 & x^5 \qquad - x^4y \qquad + x^3y^2 \qquad - x^2y^3 \qquad + xy^4 \qquad - y^5 \\
\hline
x + y & x^6 + x^5y - x^5y - x^4y^2 + x^4y^2 + x^3y^3 - x^3y^3 - x^2y^4 + x^2y^4 + xy^5 - xy^5 - y^6 \\
 & \underline{x^6 + x^5y} - \underline{x^5y - x^4y^2} + \underline{x^4y^2 + x^3y^3} - \underline{x^3y^3 - x^2y^4} + \underline{x^2y^4 + xy^5} - \underline{xy^5 - y^6} \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

Observe, (i) The natural order of the terms.

(ii) That the signs of the quotient run plus and minus, alternately.

Generally, $x^{\text{even}} - \overset{\text{same}}{y^{\text{even}}}$ is exactly divisible by $x+y$.

It is readily seen that x^6+y^6 is not exactly divisible by $x+y$.

Question: What is the remainder?

4. x^6-y^6 is exactly divisible by $x-y$.

$$\begin{array}{r} x^5 \qquad + x^4y \qquad + x^3y^2 \qquad + x^2y^3 \qquad + xy^4 \qquad + y^5 \\ \hline x-y)\,x^6 - x^5y + x^5y - x^4y^2 + x^4y^2 - x^3y^3 + x^3y^3 - x^2y^4 + x^2y^4 - xy^5 + xy^5 - y^6 \\ \underline{x^6 - x^5y} + \underline{x^5y - x^4y^2} + \underline{x^4y^2 - x^3y^3} + \underline{x^3y^3 - x^2y^4} + \underline{x^2y^4 - xy^5} + \underline{xy^5 - y^6} \\ \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \qquad \cdot \quad \cdot \end{array}$$

Observe, (i) The natural order of the terms.

(ii) That all the signs of the quotient are plus.

Generally, $x^{\text{even}} - \overset{\text{same}}{y^{\text{even}}}$ is exactly divisible by $x-y$.

It is readily seen from the above example, that x^6+y^6 is not exactly divisible by $x-y$. What is the remainder?

Summary.

The following are exactly divisible:

I. $\dfrac{x^{\text{odd}} + \overset{\text{same}}{y^{\text{odd}}}}{x+y}$. Simplest example, $\dfrac{x+y}{x+y}=1$.

II. $\dfrac{x^{\text{odd}} - \overset{\text{same}}{y^{\text{odd}}}}{x-y}$. Simplest example, $\dfrac{x-y}{x-y}=1$.

III. $\dfrac{x^{\text{even}} - \overset{\text{same}}{y^{\text{even}}}}{x+y}$. Simplest example, $\dfrac{x^2-y^2}{x+y}=x-y$.

IV. $\dfrac{x^{\text{even}} - \overset{\text{same}}{y^{\text{even}}}}{x-y}$. Simplest example, $\dfrac{x^2-y^2}{x-y}=x+y$.

Signs. When the signs between the terms are minus in both numerator and denominator, the signs of the quotient are all plus. In all other cases the signs are alternately plus and minus.

In testing whether such expressions are divisible by $x \pm y$, try the expression of the lowest odd or even power.

E.g. $x^{10}-y^{10}$ is divisible by $x+y$, because x^2-y^2 is divisible by $x+y$.

$x^7 - y^7$ is not divisible by $x+y$, because $x-y$ is not divisible by $x+y$.

Or the expressions may be tested by the substitution method given on page 127.

Thus, $x-y$ is a factor of $x^9 - y^9$, because on substituting y for x,

$$x^9 - y^9 = y^9 - y^9 = 0.$$

It is not a factor of $x^9 + y^9$, because on substituting y for x,

$$x^9 + y^9 = y^9 + y^9 = 2y^9, \text{ not } 0.$$

Exercise XII (e)

What are the factors of :

1. $x^5 + y^5$? 2. $x^5 - y^5$? 3. $x^3 - y^3$? 4. $x^6 - y^6$?
5. $x^4 - y^4$? 6. $x^9 + y^9$? 7. $x^9 - y^9$?

Write down the answers to the following:

8. $\dfrac{x^4 - y^4}{x+y}$. 9. $\dfrac{x^4 - y^4}{x-y}$. 10. $(16x^4 - 81y^4) \div (2x+3y)$.
11. $(8x^3 - 27y^3) \div (2x - 3y)$. 12. $\dfrac{x^9 - y^9}{x^3 - y^3}$.
13. $\dfrac{x^6 - y^6}{x^2 - y^2}$. 14. $\dfrac{x^6 - y^6}{x^3 - y^3}$. 15. $\dfrac{a^5 - 32b^5}{a - 2b}$.
16. Show that $x - 3$ is a factor of $x^3 - 6x^2 + 5x + 12$. Find the remaining factors.
17. State the factors of $R^3 + r^3$, and of $R^3 - r^3$.
18. Write down the answer to :
$$\frac{(a+b)^3 - 64(c-d)^3}{(a+b) - 4(c-d)}.$$
19. Simplify : $$\frac{(a-b)^4 - (b-c)^4}{a-c}.$$
20. Simplify : $$\frac{(a+b)^3 - 8(b-c)^3}{a-b+2c}.$$

§ 5. Application of Factors.

Highest Common Factors and Lowest Common Multiple of Expressions.

When given expressions can be factorised, factors common to the expressions are readily found.

Example.—Find the H.C.F. of

$$2x^3 + 6x^2 - 20x, \quad 2x^2 - 7x + 6, \quad x^2 - 7x + 10.$$

Factorising each expression, we have:

$$2x^3+6x^2-20x=2x(x-2)(x+5),$$
$$2x^2-7x+6=(x-2)(2x-3),$$
$$x^2-7x+10=(x-2)(x-5).$$

Examining these factors, it is seen that the factor $(x-2)$ is common to all the expressions. It is, moreover, the highest common factor.

I.e. the H.C.F. is $(x-2)$. The result can be checked by the method given on page 52.

EXAMPLE.—Find the L.C.M. of

$2x^3+6x^2-20x$, $2x^2-7x+6$, $x^2-7x+10$, and x^2-4x+4.

Factorising as before:

$$2x^3+6x^2-20x=2x(x-2)(x+5),$$
$$2x^2-7x+6=(x-2)(2x-3),$$
$$x^2-7x+10=(x-2)(x-5),$$
$$x^2-4x+4=(x-2)(x-2)=(x-2)^2.$$

Since the L.C.M. must contain each expression, it must necessarily contain the factors of each expression. Thus, it must contain $2x$, $(x+5)$, $(2x-3)$, $(x-5)$, and also $(x-2)^2$. If $(x-2)$ to the first power only is included, the result will not contain the whole of the expression x^2-4x+4, but only one factor of it.

The L.C.M. is $2x(x+5)(2x-3)(2x-5)(x-2)^2$.

The result can be checked by division, as shown on page 53.

§ 6. Fractions.

EXAMPLE i.—Simplify:

$$\frac{a}{a-b}-\frac{2ab}{a^2-b^2}-\frac{b}{a+b}.$$

$$\text{The given expression}=\frac{a(a+b)-2ab-b(a-b)}{\text{L.C.M. }(a+b)(a-b)}$$
$$=\frac{a^2+ab-2ab-ab+b^2}{(a+b)(a-b)}$$
$$=\frac{a^2-2ab+b^2}{(a+b)(a-b)}$$
$$=\frac{(a-b)\cancel{(a-b)}}{(a+b)\cancel{(a-b)}}$$
$$=\frac{a-b}{a+b}.$$

EXAMPLE ii.—Simplify: $\dfrac{\dfrac{x^2+x-6}{x^2-16}}{\dfrac{x^2-5x+6}{x^2+6x+8}}$.

$$\frac{\dfrac{x^2+x-6}{x^2-16}}{\dfrac{x^2-5x+6}{x^2+6x+8}}=\frac{(x+3)\cancel{(x-2)}}{\cancel{(x+4)}(x-4)}\times\frac{\cancel{(x+4)}(x+2)}{(x-3)\cancel{(x-2)}}$$
$$=\frac{(x+3)(x+2)}{(x-4)(x-3)}$$
$$=\frac{x^2+5x+6}{x^2-7x+12}.$$

EXAMPLE iii.—Solve: $\dfrac{5}{x-a}-\dfrac{3}{x-b}=\dfrac{2}{x}$.

Multiply both sides of the equation by $x(x-a)(x-b)$, the L.C.M. of the denominators:

$$5x(x-b)-3x(x-a)=2(x-a)(x-b),$$
$$5x^2-5bx-3x^2+3ax=2x^2-2ax-2bx+2ab.$$

Arrange all terms in x^2 and in x on one side:

$$5x^2-3x^2-2x^2+3ax+2ax-5bx+2bx=2ab.$$

Observe that x^2 vanishes:

$$5ax-3bx=2ab,$$
$$x(5a-3b)=2ab,$$
$$x=\frac{2ab}{5a-3b}.$$

EXERCISE XII (F)

Find the H.C.F. and L.C.M. of:

1. x^2-3x+4 and x^2-5x+4.
2. x^2+2x-3, x^3-x^2-12x.
3. $2x^2-4x-6$ and $4x^2$.
4. $(x+2)(x^2-x-2)$ and x^3-x^2-4x+4.
5. $6a^2+a-2$, $3a^2+5a+2$, $3a^3-a^2+a+2$.
6. a^2-b^2, $a^2-3ab+2b^2$, $a^2-2ab+b^2$.
7. a^3-b^3, a^2+ab+b^2, a^3+b^3.
8. Show that: $1-\dfrac{b}{a+b}=\dfrac{a}{a+b}$.

Reduce to its lowest terms:

9. $\dfrac{x^2-3x-4}{x^2-5x+4}$. (Factorise the numerator and denominator.)

10. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$.

11. $\dfrac{24x^4-22x^2+5}{48x^4+16x^2-15}$.

12. Simplify: $\dfrac{x^2-9x+20}{x^2-6x} \times \dfrac{x^2-13x+42}{x^2-5x}$.

13. Divide: $\dfrac{x^2-5x+6}{x^2-5x}$ by $\dfrac{x^2-3x}{x^2-6x+5}$.

Simplify:

14. $\left(\dfrac{x-b}{a-b}+\dfrac{x-c}{c-a}\right) \div (c-b)$.

15. $\dfrac{1}{x-2}-\dfrac{1}{x+2}-\dfrac{6}{x^2-4}$.

16. $\dfrac{2a+3}{a-a^2}+\dfrac{5-a}{a-1}-\dfrac{3}{a}$.

17. $\dfrac{1}{x+2y}+\dfrac{3y}{x^2+xy-2y^2}+\dfrac{y}{(x-y)^2}$.

18. $\left(a+1+\dfrac{8}{a-5}\right)\left(a-1-\dfrac{8}{a-3}\right)$.

19. Show that $\dfrac{a}{1-ax}+\dfrac{b}{1-bx}$ has the same value when $x=\dfrac{1}{a}+\dfrac{1}{b}$ as when $x=\dfrac{2}{a+b}$.

Now write down the values of x which satisfy the equation:

$$\frac{a}{1-ax}+\frac{b}{1-bx}+a+b=0.$$

Simplify:

20. $\dfrac{4x-7}{x-1}-\dfrac{8x-14}{x+1}-\dfrac{4x-7}{x^2-1}$.

21. $\dfrac{ax}{a-b}-x$.

22. (i) $\dfrac{1}{x-y}+\dfrac{2y}{x^2-y^2}$. (ii) $\dfrac{x-3}{x^2+5x+6}-\dfrac{x+3}{x^2-x-6}$.

23. (i) $\dfrac{a^3}{(a-b)(b-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

(ii) $\sum\limits_{abc}\dfrac{a+b}{(b-c)(c-a)}$.

24. Find : $\dfrac{x^2-2x+1}{x^2-5x+6} \times \dfrac{x^2-4x+4}{x^2-4x+3} \times \dfrac{x^2-6x+9}{x^2-3x+2}$.

25. Divide $\dfrac{a}{a+b} + \dfrac{b}{a-b}$ by $\dfrac{a^2}{a^2-b^2} - \dfrac{b^2}{a^2+b^2}$.

26. From $\dfrac{xy}{x^2+2xy+y^2}$ take $\dfrac{y}{x+y}$.

27. Simplify : $\dfrac{a^2-b^2}{2(a+b)} - \dfrac{a^2+b^2}{2a-2b} + \dfrac{2a^2b+2ab^2}{a^2-b^2}$.

Solve :

28. $12 - \dfrac{5x-10}{7x} = \dfrac{35}{x} - 22\tfrac{2}{7}$.

29. $\dfrac{1}{x-1} + \dfrac{2}{x-3} = \dfrac{3}{x-2}$.

30. $\dfrac{ax}{a+b} - x = \dfrac{b^2x}{a^2-b^2} - \dfrac{ab}{a-b}$.

31. Find A and B such that :

$$\frac{12x-5}{6x^2-5x-6} = \frac{A}{3x+2} + \frac{B}{2x-3}.$$ (*See Ex. XI* (B), *29.*)

CHAPTER XIII

SURDS

§ **1.** There are some roots which cannot be determined exactly, and which are therefore most conveniently treated as algebraic numbers.

EXAMPLES.—$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, etc.

If an attempt is made to extract these roots, it will be found that their decimal portion neither terminates nor recurs. Neither can the roots be expressed in vulgar fraction form. They are said to be **Irrational.**

Such roots are called **Surds.**

EXERCISE XIII (A)

Calculate correct to five decimal places :

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$.

§ 2. Fundamental Examples.

i. Addition and subtraction of unlike surds.

$\sqrt{2}$ added to $\sqrt{5} = \sqrt{5} + \sqrt{2}$. $\sqrt{2}$ subtracted from $\sqrt{5} = \sqrt{5} - \sqrt{2}$.

The above are similar to adding and subtracting a and b.

Using the numbers obtained in Exercise XIII (A), the pupil should convince himself that $\sqrt{5} + \sqrt{2}$ does not equal $\sqrt{7}$, and that $\sqrt{5} - \sqrt{2}$ does not equal $\sqrt{3}$.

ii. Addition and subtraction of like surds.

$\sqrt{2}$ added to $\sqrt{2} = 2\sqrt{2}$. Compare, $a + a = 2a$.

$3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$. Compare, $3a + 2a = 5a$.

$5\sqrt{2} - 2\sqrt{2} = 3\sqrt{2}$. Compare, $5a - 2a = 3a$.

Remember that in $3\sqrt{2}$, the figure 3 is a coefficient.

iii. Products and quotients of unlike surds.

$\sqrt{3} \times \sqrt{2} = \sqrt{3 \times 2} = \sqrt{6}$, $2\sqrt{3} \times 5\sqrt{2} = 10\sqrt{6}$,

$\sqrt{6} \div \sqrt{2} = \sqrt{\frac{6}{2}} = \sqrt{3}$, $\frac{10\sqrt{6}}{2\sqrt{3}} = \frac{\overset{5}{\cancel{10}} \times \overset{\sqrt{2}}{\cancel{\sqrt{6}}}}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{\sqrt{3}}}} = 5\sqrt{2}$.

iv. Products and quotients of like surds.

$\sqrt{2} \times \sqrt{2} = 2$, $\sqrt{3} \times \sqrt{3} = 3$,

$3\sqrt{2} \times 2\sqrt{2} = 3 \times 2 \times \sqrt{2} \times \sqrt{2} = 6 \times 2 = 12$,

$$\frac{6\sqrt{2}}{2\sqrt{2}} = \frac{\overset{3}{\cancel{6}} \times \overset{1}{\cancel{\sqrt{2}}}}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{\sqrt{2}}}} = 3.$$

v. Simplification of surds.

Sometimes a given surd has a factor which is rational. In such a case, the factor can be placed as a coefficient, as shown in the following examples:

$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$. $\sqrt{243} = \sqrt{81 \times 3} = \sqrt{81} \times \sqrt{3} = 9\sqrt{3}$.

Exercise XIII (B)

1. Verify the foregoing examples by making use of the numbers found in the last exercise.

2. Express in simpler form:

$$\sqrt{27}, \quad \sqrt{48}, \quad \sqrt{125}, \quad \sqrt{252}.$$

3. Given that $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$, $\sqrt{5}=2{\cdot}236$, find $\sqrt{8}$, $\sqrt{27}$, $\sqrt{48}$, $\sqrt{12}$, $\sqrt{45}$.

4. Simplify: (i) $3\sqrt{2}-5\sqrt{3}+6\sqrt{2}+\sqrt{3}-\sqrt{27}$.
   (ii) $\sqrt{7}+3\sqrt{3}-\sqrt{245}+\sqrt{12}-\sqrt{63}+\sqrt{45}+\sqrt{28}$.

5. (i) $10\sqrt{3}\times 2\sqrt{3}=\quad ?$ (ii) $\sqrt{2}\times\sqrt{3}\times\sqrt{5}=\quad ?$
   (iii) $(\sqrt{3}+\sqrt{2})\times\sqrt{3}=\quad ?$ (iv) $\dfrac{2\sqrt{3}}{3\sqrt{2}}\times 6\sqrt{6}=\quad ?$

6. $(2\sqrt{3}+5\sqrt{2})(3\sqrt{3}-2\sqrt{2})=\quad ?$ Compare $(2a+5b)(3a-2b)$.

7. Expand $(\sqrt{a}+\sqrt{b})^2$ and $(\sqrt{a}-\sqrt{b})^2$.

8. (i) Add $3\sqrt{5}$ to $3\sqrt{2}$. (ii) From $8\sqrt{3}$ take $3\sqrt{3}$.

9. To $6\sqrt{5}$ add $\sqrt{125}$.

10. (i) Add together $\sqrt{3}-1$ and $\sqrt{3}+1$.
    (ii) Find the difference between $\sqrt{3}-1$ and $\sqrt{3-1}$.

11. Evaluate: $3(\sqrt{3}+\sqrt{2})-\sqrt{2}(2-\sqrt{2})+\sqrt{3}(3-\sqrt{3})$.

12. (i) Find the difference between $10\sqrt{{\cdot}2}$ and $\sqrt{2}$, $\sqrt{2}$ and $10\sqrt{{\cdot}02}$.
    (ii) Determine in decimal form:
    $$\sqrt{{\cdot}02},\quad \sqrt{{\cdot}2},\quad \sqrt{{\cdot}03}+\sqrt{{\cdot}02},\quad \left(\frac{\sqrt{{\cdot}03}}{\sqrt{{\cdot}02}}-\sqrt{{\cdot}03}\times\sqrt{{\cdot}02}\right).$$

13. Solve the equation, $3\sqrt{x}+2\sqrt{3}=\sqrt{2x}-\sqrt{2}$.

14. What are the factors of x^2-3 and of $2x^2-3$?

§3. Applications.

To evaluate $\dfrac{5}{\sqrt{3}}$.

It is advisable to avoid surds as divisors.

In order to remove $\sqrt{3}$ from the denominator, multiply both numerator and denominator by $\sqrt{3}$. Thus:

$$\frac{5}{\sqrt{3}}=\frac{5\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{5\sqrt{3}}{3}=\frac{5\times 1{\cdot}732}{3}=2{\cdot}8866\ldots=2{\cdot}887 \text{ (approx.)}.$$

This process is known as **Rationalisation** of the denominator.

§4. Conjugate Surds.

The following example is very important:

$$\begin{aligned}(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})&=(\sqrt{3})^2-(\sqrt{2})^2\\&=3-2\\&=\mathbf{1}.\end{aligned}$$

Compare this with the general example:

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b.$$

Observe that no surd appears in the product.

The sum of, and the difference between two surds are said to be conjugate.

The use of conjugate surds is shown in the following example.

Evaluate: $\dfrac{5\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$.

Multiply the numerator and the denominator by the conjugate of the denominator. Then:

$$\frac{5\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}=\frac{5\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}\times\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}=\frac{30+21\sqrt{6}+18}{(3\sqrt{2})^2-(2\sqrt{3})^2}$$

$$=\frac{48+21\sqrt{6}}{18-12}$$

$$=\frac{48+21\sqrt{6}}{6}$$

$$=8+\tfrac{7}{2}\sqrt{6}.$$

Observe that the final result is much simpler than the original expression, and that to obtain the answer no division by a decimal is necessary.

Complete the computation.

Exercise XIII (c)

Calculate in the shortest manner:

1. $\dfrac{2}{\sqrt{5}}$, $\dfrac{1}{\sqrt{3}}$, $\dfrac{6}{\sqrt{7}}$, $\sqrt{\dfrac{2}{3}}$.

2. $\dfrac{1}{\sqrt{2}-1}$, $\dfrac{1}{1+\sqrt{2}}$, $\dfrac{\sqrt{2}}{1+\sqrt{2}}$.

3. $\dfrac{\sqrt{2}}{3-\sqrt{2}}$, $\dfrac{1+\sqrt{2}}{\sqrt{2}-1}$.

4. Find the value of $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}$,

   when $a=\sqrt{3}$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$.

5. Simplify: $\dfrac{\sqrt{2}}{2}-\dfrac{1}{\sqrt{2}}$.

6. Multiply: (i) $(2\sqrt{3}+3\sqrt{2})$ by $(2\sqrt{3}-3\sqrt{2})$.

   (ii) $(a\sqrt{3}+b\sqrt{2})$ by $(a\sqrt{3}-b\sqrt{2})$.

   (iii) $\left(\dfrac{2}{\sqrt{3}}+\dfrac{3}{\sqrt{2}}\right)$ by $\left(\dfrac{2}{\sqrt{3}}-\dfrac{3}{\sqrt{2}}\right)$.

7. What is the value of $x^2 - \frac{1}{x^2}$, when $x = 2 + \sqrt{3}$?

8. Rationalise the denominators, and evaluate the following:

(i) $\frac{2+\sqrt{2}}{2-\sqrt{2}}$. (ii) $\frac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

(iii) $\frac{2\sqrt{5}+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$. (iv) $\frac{2\sqrt{3}+3\sqrt{2}}{3\sqrt{5}-2\sqrt{3}}$.

9. Find the value of $(2\sqrt{2}+\sqrt{3})(\sqrt{6}-4)$.

10. Simplify: $\frac{7+\sqrt{21}}{7-\sqrt{21}} + \frac{\sqrt{21}-3}{\sqrt{21}+3}$. 11. Calculate: $\frac{(\sqrt{3}-\sqrt{2})^2}{2-\sqrt{2}}$.

12. Find the value of $\frac{x+y}{x-y} + \frac{x-y}{x+y}$,

when $x = \sqrt{3+\sqrt{2}}$ and $y = \sqrt{3-\sqrt{2}}$.

§ **5.** If the side of a square is a and the diagonal d, then

$$d = a\sqrt{2}.$$

If it is required to find the side in terms of the diagonal, then $a = \frac{d}{\sqrt{2}}$, which, when rationalised, becomes

$$a = \frac{d\sqrt{2}}{2}.$$

§ **6.** If one of the angles of an equilateral triangle is bisected, and the bisector produced to meet the opposite side, then each part is a triangle with its angles 60°, 30° and 90° respectively. In fig. 1, let BC be of unit length. Then:

AB is 2 units and AC is $\sqrt{2^2 - 1^2} = \sqrt{3}$ units.

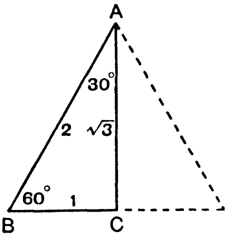


FIG. 1.

It follows that all triangles similar to this have their sides in the ratio of $1 : 2 : \sqrt{3}$, or, in order of size, $1 : \sqrt{3} : 2$.

EXAMPLE i.—In a given 60°, 30°, right-angled triangle, the shortest side is 10 cms. Find the remaining sides.

$$\frac{AB}{BC}=\frac{2}{1},\qquad AB=2BC;\qquad \therefore\ AB=20 \text{ cms.}$$

$$\frac{AC}{BC}=\frac{\sqrt{3}}{1}.\qquad AC=BC\sqrt{3};\qquad \therefore\ AC=10\sqrt{3} \text{ cms.}$$

EXAMPLE ii.—In a given 60°, 30°, right-angled triangle, the side opposite the angle 60° is 10 inches. Find the remaining sides.

$$\frac{BC}{AC}=\frac{1}{\sqrt{3}}$$

$$BC=\frac{AC}{\sqrt{3}};$$

$$\therefore\ BC=\frac{10}{\sqrt{3}},$$

$$\text{i.e.}\quad BC=\frac{10\sqrt{3}}{3} \text{ inches.}$$

$$\frac{AB}{BC}=\frac{2}{1},$$

$$AB=2BC;$$

$$\therefore\ AB=\frac{20\sqrt{3}}{3} \text{ inches.}$$

EXERCISE XIII (D)

1. Find the ratio of the sides of a right-angled isosceles triangle.

2. By means of a right-angled triangle, find a straight line equal to $\sqrt{a^2+\frac{a^2}{4}}$, when a is the length of a given straight line.

3. The hypotenuse of a 60°, 30°, right-angled triangle is 10 cms. Find the remaining sides.

4. The diagonal of a square courtyard measures 40 yards. Find its area and the length of its sides.

5. Find in surd form the trigonometrical ratios of 30°, 45° and 60°.

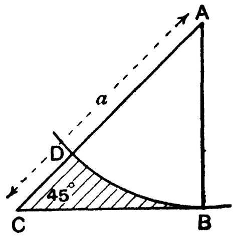


FIG. 2.

6. In fig. 2, find the length of CD and the shaded area CDB, in terms of a.

7. In fig. 3, find CD and the area of the triangle CDB, in terms of x.

8. In fig. 4, find CD, and the shaded area, in terms of $AB = x$.

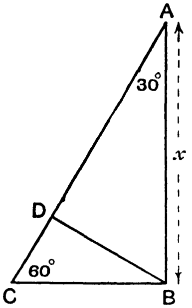


FIG. 3.

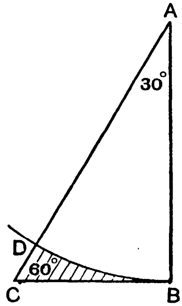


FIG. 4.

9. The slant edges of a square pyramid make angles of 60° with the base. Calculate their length if the side of the base measures 5 inches.

10. The usual way of fitting a circular filter paper is to fold it into quadrants and then to open it in the form of a cone. Show that the apex angle of a vertical section of this cone through the apex is 60°.

If the diameter of the paper is 5 inches, find the altitude of the cone.

11. The area of a triangle, the sides of which measure a, b and c units, is $\sqrt{s(s-a)(s-b)(s-c)}$, where s stands for the semi-sum of a, b and c. Determine the area of the triangle, the sides of which are 6, 8 and 10 inches.

12. Apply the formula given in Exercise 11, for the area of a triangle, to the case of an equilateral triangle of side a, and show that it agrees with the rule for calculating the area of a triangle from the base and altitude.

CHAPTER XIV

LOGARITHMS, THE SLIDE RULE

§ 1. Logarithms.

It has been mentioned on page 37, that logarithm is another name for index. More particularly, the logarithm of a number is the index of the power to which another number, called the base, must be raised to equal the given number.

When 10 is chosen as the base, the logarithms are called **common** logarithms.

Number	·01	·1	1	10	100	1000	10000
log base 10	−2	−1	0	1	2	3	4

Examining the above table, starting from the right and working towards the left, you notice that as the numbers are divided by 10, the logarithms decrease by 1. It follows, naturally,

that the log of 1 to the base 10 is 0, i.e. $\log_{10}1 = 0$,
" " ·1 " " -1, " $\log_{10}{\cdot}1 = -1$,
" " ·01 " " -2, " $\log_{10}{\cdot}01 = -2$.

Let us prove this:

$1 = \frac{10}{10}$, i.e. $\frac{10^1}{10^1} = 10^{1-1} = 10^0$, since we subtract indices when dividing powers of the same base;

$$\therefore \log_{10}1 = 0.*$$

Similarly, $$\cdot 1 = \frac{1}{10} = \frac{10^0}{10^1} = 10^{0-1} = 10^{-1};$$

$$\therefore \log_{10}{\cdot}1 = -1.$$

EXAMPLE.—In the same way, show that $\log_{10}{\cdot}01 = -2$, and $\log_{10}{\cdot}0001 = -4$.

It follows from the table that

(i) The logarithms of numbers greater than 1 are positive.
(ii) The logarithms of numbers less than 1 are negative.

* Similarly, the log of 1 to any base is 0, for $b^0 = 1$.

§ **2.** Now consider numbers between these powers of 10.

Take the number 35, it lies between 10 and 100. Its logarithm is between 1 and 2, i.e. it is 1 plus a decimal.

Similarly, 350 lies between 100 and 1000, and its logarithm is between 2 and 3, and ·08 lies between ·01 and ·1, and therefore its logarithm is between − 2 and − 1.

QUESTIONS.

Between what numbers are the logarithms of

2675, 3, ·6, ·06, ·0006?

§ **3.** The graph shows the logarithms of numbers between 0 and 10.

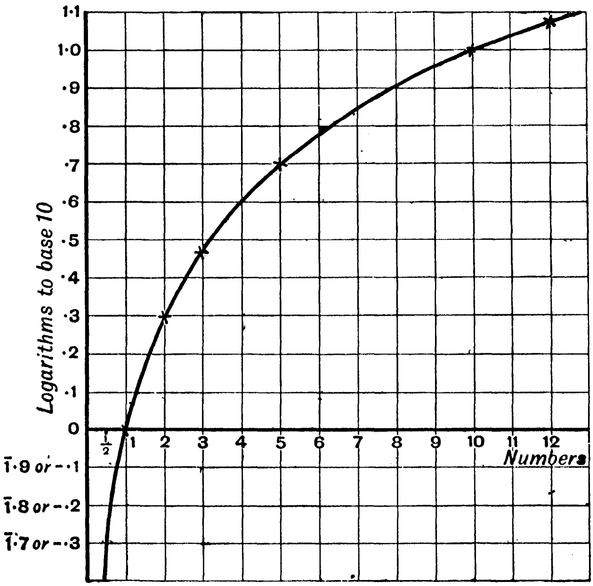


FIG. 1.

From it you will see that the log of 2 is ·3 approx.,
of 5 „ ·7 „
and of 6·3 „ ·8 „

§ **4.** With these numbers we can illustrate several important uses of logarithms.

(i) To find log 20.

$$\begin{aligned} 20 &= 10 \times 2 \\ &= 10^1 \times 10^{0.3} \quad (\log_{10} 2 = \cdot 3, \text{ i.e. } 10^{0.3} = 2) \\ &= 10^{1.3}, \quad \text{adding indices,} \end{aligned}$$

i.e. $\log_{10} 20 = 1{\cdot}3$.

(ii) To find log 63.

$$\begin{aligned} 63 &= 10 \times 6{\cdot}3 \\ &= 10^1 \times 10^{\cdot 8} \\ &= 10^{1.8}, \end{aligned}$$

i.e. $\log_{10} 63 = 1{\cdot}8$.

(iii) To find log 63000.

$$\begin{aligned} 63000 &= 6{\cdot}3 \times 10000 \\ &= 10^{\cdot 8} \times 10^4 \\ &= 10^{4.8}; \end{aligned}$$

$\therefore \log_{10} 63000 = 4{\cdot}8$.

§ **5.** You will notice that the decimal part of the logarithm, viz. ·8, is the same for 6·3, 63, 630, 6300, 63000, etc., i.e. *It does not change with the change of the position of the decimal point in the number.*

§ **6.** The whole number part of the logarithm, however, does depend upon the position of the decimal point. Thus, for 6·3 it is 0; for 63, 1; for 630, 2; 6300, 3; etc.

§ **7.** The name given to the decimal portion of a logarithm is **mantissa**, meaning right (on the right of the point), and that given to the whole number, **characteristic**.

Hence we see that the **characteristic** depends upon the position of the decimal point. Thus, in the numbers given, it is one less than the number of figures before the decimal point.

Number.	Number of figures before the decimal point.	Characteristic.
6·3	1	0
63·	2	1
630·	3	2
6300·	4	3
63000·	5	4
	etc.	

EXERCISE (*Oral*)

1. $\log 2 = \cdot 3$ approx. State the logs of 20, 200, 2000, 20000, 200000, 2000000.

2. $\log 5 = \cdot 7$ approx. State the logs of 50, 500, 5000, 50000, 500000, 5000000.

§**8.** Let us see for what other numbers we can find the logs from those of 2, 5 and 6·3

(i) $4 = 2 \times 2$
$= 10^{0\cdot 3} \times 10^{0\cdot 3}$
$= 10^{0\cdot 6}$,
i.e. $\log 4 = 0\cdot 6$.

(ii) $12\cdot 6 = 2 \times 6\cdot 3$
$= 10^{0\cdot 3} \times 10^{\cdot 8}$
$= 10^{1\cdot 1}$,
i.e. $\log 12\cdot 6 = 1\cdot 1$.
Notice that the characteristic is correct.

(iii) $2\cdot 5 = \frac{5}{2}$
$= \frac{10^{\cdot 7}}{10^{\cdot 3}}$
$= 10^{\cdot 4}$.
$\log 2\cdot 5 = \cdot 4$,

(iv) $1\cdot 26 = \frac{6\cdot 3}{5}$
$= \frac{10^{\cdot 8}}{10^{\cdot 7}}$ Subtracting indices (division)
$= 10^{\cdot 1}$,
i.e. $\log 1\cdot 26 = \cdot 1$.
Compare this mantissa with that of $\log 12\cdot 6$.

EXERCISE XIV (A)

1. Similarly, find $\log 25$, $\log (5)^3$, i.e. $\log 125$, $\log (2)^4$. Can you state the rule for finding the log of a power of a number from the log of the number?

2. Write down $\log 40$, $\log 4000$ and $\log \frac{1\cdot 26}{2}$.

3. Find the log of (5×2), and of $(6\cdot 3 \times 5)$. Can you write down the rule for finding the log of the product of given numbers from the logs of the numbers?

4. Write down log 1·26, 126, 1260.

5. Find the log of $\frac{6\cdot 3}{2}, \frac{63}{5}, \frac{10}{6\cdot 3}, \frac{250}{63}, \frac{625}{8}$.

Can you write down the rule for finding the log of a quotient from the logs of the numbers?

6. Write down log 25, 250, 25000.

By referring to the graph, verify that the mantissa of each of your results is approximately correct.

§ 9. Roots as Powers.

We have seen on page 36 that $\sqrt{a} \times \sqrt{a} = a$ or a^1.

Now, if we represent $\sqrt{a}$ as a to some power, the index denoting the power will be such that the sum of two such indices is 1. It follows that each is $\frac{1}{2}$. In other words, $\sqrt{a}$ equals $a^{\frac{1}{2}}$.

Another method of obtaining the same result is as follows:

Let $\sqrt{a} = a^x$; then $\sqrt{a} \times \sqrt{a} = a^1$,

i.e. $a^x \times a^x = a^1$.

But $a^x \times a^x = a^{(x+x)}$, or a^{2x};

$\therefore\ 2x = 1$ and $x = \frac{1}{2}$.

Similarly, $\sqrt[3]{a} = a^{\frac{1}{3}}$, since $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a^1$

and $\sqrt{a^3} = a^{\frac{3}{2}}$.

Application to Logarithms.

(i) To find $\log_{10}\sqrt{10}$.

$$\sqrt{10} = 10^{\frac{1}{2}}; \quad \therefore\ \log_{10}\sqrt{10} = \tfrac{1}{2} \text{ or } \cdot 5.$$

Now $\sqrt{10} = 3{\cdot}162$; $\therefore\ \log_{10}3{\cdot}162 = 0{\cdot}5$.

(ii) To find $\log_{10}\sqrt{2}$.

From the graph, $\log_{10}2 = \cdot 3$,

i.e. $10^{\cdot 3} = 2$,

or $2 = 10^{\cdot 3}$;

$$\therefore\ \sqrt{2} \text{ or } 2^{\frac{1}{2}} = \sqrt{10^{\cdot 3}} = 10^{\frac{\cdot 3}{2}} = 10^{\cdot 15},$$

i.e. $\log_{10}\sqrt{2} = \cdot 15$.

Now $\sqrt{2} = 1{\cdot}414$.

Hence $\log_{10}1{\cdot}414 = \cdot 15$.

It will be observed that $\log_{10}\sqrt{2} = \frac{1}{2}\log_{10}2$.

(iii) To find $\log_{10}\sqrt[3]{5}$.

$$\sqrt[3]{5} = \sqrt[3]{10^{\cdot 7}} \quad (\text{since } \log_{10}5 = \cdot 7)$$

$$= 10^{\frac{\cdot 7}{3}}.$$

Hence $\log_{10}\sqrt[3]{5} = \dfrac{\cdot 7}{3}$, i.e. $\dfrac{1}{3}\log_{10}5$.

§ 10. Summary.

(i) The logarithm of a product is obtained by adding the logarithms of the factors,

$$\text{i.e.}\quad \log(xy) = \log x + \log y.$$

(ii) The logarithm of a quotient is obtained by subtracting the logarithm of the divisor from that of the dividend. Thus:

$$\log\frac{x}{y} = \log x - \log y.$$

(iii) The logarithm of a power of a number is obtained by multiplying the logarithm of the number by the index of the power. Thus:

$$\log(x^n) = n\log x,$$

$$\log\sqrt[n]{x} = \log\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log x.$$

Observe that a root is regarded as a fractional power.

Exercise XIV (b)

1. Express as powers, without root signs:

$$\sqrt[3]{a^5},\quad \sqrt{a^7},\quad \sqrt[4]{a^2},\quad \sqrt[5]{a},\quad \sqrt{(a+b)^3}.$$

2. Multiply $\left(a^{\frac{1}{3}} + b^{\frac{1}{2}}\right)$ by $\left(a^{\frac{2}{3}} - b^{\frac{1}{4}}\right)$.
3. Divide $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{4}} + b^{\frac{1}{4}}$.
4. Find $\log_{10}\sqrt{6{\cdot}3}$, $\log_{10}\sqrt[3]{6{\cdot}3}$, $\log_{10}\sqrt[3]{2}$.
5. Determine $\log_{10}\sqrt{2^3}$, $\log_{10}\sqrt[3]{2^2}$, $\log_{10}\sqrt[3]{5^2}$.
6. From $\log_{10}2$ and $\log_{10}3$, determine:

$$\log_{10}6,\quad \log_{10}1{\cdot}5,\quad \log_{10}15,\quad \log_{10}8,\quad \log_{10}9,$$

$$\log_{10}27,\quad \log_{10}2{\cdot}25,\quad \log_{10}\frac{10}{2},\quad \log_{10}\frac{10}{3},\quad \log_{10}33{\cdot}\dot{3}.$$

§ 11. The Logarithm of a Pure Decimal.

(i) To find $\log 0{\cdot}2$.

$$0{\cdot}2 = \tfrac{2}{10};$$

$$\therefore\ \log 0{\cdot}2 = \log 2 - \log 10$$

$$= {\cdot}3 - 1.$$

Now $0{\cdot}3 - 1 = -0{\cdot}7$, but for a reason to be explained later, it is not usual to write it in this form, but as $\bar{1}{\cdot}3$.

The minus sign is placed over the characteristic, and the mantissa remains positive.

Observe that the mantissa of $\log_{10} 0\cdot 2$ is the same as that of $\log_{10} 2$.

(ii) To find $\log_{10} 0\cdot 005$.

$$\begin{aligned} 0\cdot 005 &= \tfrac{5}{1000}; \\ \therefore\ \log 0\cdot 005 &= \log 5 - \log 1000 \\ &= 0\cdot 7 - 3 \\ &= \bar{3}\cdot 7. \end{aligned}$$

Similarly, verify that $\log 0\cdot 5$ is $\bar{1}\cdot 7$, $\log 0\cdot 05$, $\bar{2}\cdot 7$, etc.

It is seen that the characteristic of the logarithm of a pure decimal (i.e. a number less than unity) is negative, and is numerically one more than the number of noughts immediately after the decimal point. Thus:

Number.	Number of noughts directly after the decimal point.	Characteristic of log.
·5	0	$\bar{1}$
·05	1	$\bar{2}$
·005	2	$\bar{3}$
·0005	3	$\bar{4}$
	etc.	

Remember that the **mantissa** of a logarithm is usually positive.

§ 12. Operations involving Negative Characteristics.

(i) To find the logarithm of $(0\cdot 2 \times 6\cdot 3)$.

$$\begin{aligned} \log (0\cdot 2 \times 6\cdot 3) &= \log 0\cdot 2 + \log 6\cdot 3 \\ &= \bar{1}\cdot 3 + 0\cdot 8 \\ &= \begin{array}{r} \bar{1}\cdot 3 \\ +0\cdot 8 \\ \hline 0\cdot 1 \end{array} \left\{ \begin{array}{l} \text{Adding the mantissae, we get} \\ \quad 1\cdot 1, \text{ which is all positive.} \\ \text{Then } \bar{1}+1=0. \end{array} \right. \end{aligned}$$

Remember that numbers carried forward from the mantissae are positive.

(ii) To find $\log \dfrac{6\cdot 3}{0\cdot 2}$.

$$\begin{aligned} \log \frac{6\cdot 3}{0\cdot 2} &= \log 6\cdot 3 - \log 0\cdot 2 \\ &= \begin{array}{r} 0\cdot 8 \\ -\bar{1}\cdot 3 \\ \hline 1\cdot 5 \end{array} \left\{ \begin{array}{l} \text{Subtracting the mantissae, we} \\ \quad \text{get } \cdot 5. \\ \text{Subtracting } \bar{1} \text{ from } 0, \text{ we get} +1. \end{array} \right. \end{aligned}$$

(iii) To find $\log \frac{0{\cdot}05}{0{\cdot}063}$.

$$\log \frac{0{\cdot}05}{0{\cdot}063} = \log 0{\cdot}05 - \log 0{\cdot}063$$

$$\begin{array}{r} = \bar{2}{\cdot}7 \\ -\bar{2}{\cdot}8 \\ \hline \bar{1}{\cdot}9 \end{array}$$

Subtracting ·8 from 1·7, we get ·9. Pay back 1 to $\bar{2}$ and we get $\bar{1}$ to subtract from $\bar{2}$, which gives $\bar{1}$.

The operation is actually:

$$\begin{array}{rr} & \bar{2}+1{\cdot}7 \\ \text{Subtract,} & (\bar{2}+1)+\ \ {\cdot}8 \\ \hline & \bar{1}+\ \ {\cdot}9 \end{array}$$

Or the operation may be carried out from the equivalent form:

$$\begin{array}{r} \bar{3}+1{\cdot}7 \\ -(\bar{2}+\ \ {\cdot}8) \\ \hline \bar{1}+\ \ {\cdot}9 \end{array}$$

(iv) To find $\log(0{\cdot}05)^2$.

$$\begin{aligned} \log(0{\cdot}05)^2 &= 2\log 0{\cdot}05 \\ &= 2\times\bar{2}{\cdot}7 \\ &= \bar{3}{\cdot}4 \end{aligned}$$

Multiply ·7 by 2; result, $+1{\cdot}4$.
Multiply $\bar{2}$ by 2; result, $\bar{4}$.
$\bar{4}+1{\cdot}4=\bar{3}{\cdot}4$.

(v) To find $\sqrt[3]{0{\cdot}02}$.

$$\log\sqrt[3]{0{\cdot}02} = \tfrac{1}{3}\log 0{\cdot}02 = \tfrac{1}{3}\times\bar{2}{\cdot}3.$$

When the negative characteristic is not exactly divisible, the operation is most conveniently carried out by changing the characteristic to the next divisible characteristic, in this case $\bar{3}$, and adding a corresponding positive number, in this case 1, to counterbalance the change.

Thus: $\tfrac{1}{3}\times\bar{2}{\cdot}3 = \tfrac{1}{3}(\bar{3}+1{\cdot}3) = \bar{1}{\cdot}4\dot{3}$.

(vi) To find $\frac{1}{7}$ of $\overline{16}{\cdot}6$.

$$\tfrac{1}{7}\times\overline{16}{\cdot}6 = \tfrac{1}{7}(\overline{21}+5{\cdot}6) = \bar{3}{\cdot}8.$$

Exercise XIV (c)

1. Write down the characteristics of the logarithms of the following numbers: 0·063, 0·00063, 0·00505, 0·5, 0·006003.

2. How many noughts are there immediately after the decimal point in numbers the logarithms of which have the following characteristics:

$$\bar{1},\quad \bar{4},\quad \bar{2},\quad \bar{6},\quad \bar{3},\quad \bar{8}?$$

3. Add: $\bar{2}{\cdot}6275$ and $3{\cdot}864$, $\bar{1}{\cdot}064$ and $1{\cdot}952$,
$\bar{6}{\cdot}3075$ and $4{\cdot}893$, $\bar{3}{\cdot}826$ and $2{\cdot}473$.

4. From $0{\cdot}76$ subtract $\bar{1}{\cdot}28$. From $\bar{1}{\cdot}28$ subtract $0{\cdot}76$.
From $\bar{2}{\cdot}064$ subtract $1{\cdot}83$. From $\bar{1}{\cdot}125$ subtract $\bar{2}{\cdot}941$.

5. Multiply $\bar{4}{\cdot}842$ by 5, and $\bar{2}{\cdot}634$ by 10.

6. Divide $\bar{4}{\cdot}842$ by 5, $\overline{12}{\cdot}1$ by 7, and $\bar{1}{\cdot}61$ by 3.

§ 13. Logarithm Tables.

So far we have used convenient numbers only, and their logarithms to the first place of decimals.

It is explained in a later chapter how logarithms may be calculated. Tables of the logarithms of numbers can be purchased for one penny from Messrs. Wyman & Sons, and there is no reason why you should not learn to use them.

The following are extracts from the tables:

MATHEMATICAL TABLES.

LOGARITHMS.

First 2 Figures.	Third Figure.										Fourth Figure.								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 12	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	18 17	21 20	24 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 5	9 8	11 11	14 14	17 16	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	16 15	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 19	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19

The tables do not give the characteristic of a logarithm, but the **mantissa** only.

Look at line 17. From this line, the mantissae of the logarithms of numbers of four digits, the first two of which are 1 and 7, are obtained.

Thus:

(i) For log 17, we take the number next to 17, viz. 2304.

At the head of the column in which this number is found, is the figure 0. You will remember that the mantissa of log 17 is the same as for log 170 and for log 0·17, etc.

The mantissa, being a decimal, is really ·2304.

The characteristic you must determine for yourself.

E.g. $\log 17 = 1{\cdot}2304$, $\log 1{\cdot}7 = 0{\cdot}2304$, $\log 1700 = 3{\cdot}2304$,
$\log 0{\cdot}0017 = \bar{3}{\cdot}2304$, $\log 0{\cdot}17 = \bar{1}{\cdot}2304$, etc.

(ii) If we require log 173, then we take the number on the 17 line, which is in the column headed 3. The number is 2380.

Then $\log 173 = 2{\cdot}2380$, $\log 1{\cdot}73 = 0{\cdot}2380$,
$\log 0{\cdot}0173 = \bar{2}{\cdot}2380$, etc.

(iii) If the number has a fourth figure, and is, say, 1738, then having got the log of 173, we move along the 17 line until we reach the next set of figures having at the head of the columns the numbers 1 to 9. These numbers have to be added to those in the other columns. Thus, in the 17 line, in the column headed 8, the number 20 is found. Adding this number to 2380, the mantissa of log 1738 is obtained, namely 2400.

You will observe that in some cases there are for the same horizontal line two sets of fourth-figure numbers. This is because the numbers to be added depend upon the third digit of the number for which the logarithm is required.

E.g. for the mantissa of 1768 we take 2455 and add 19, not 20, obtaining 2474.

(iv) For single numbers such as 5, 7, etc., use the lines 50, 70, etc., since the mantissae of the logarithms of these tens are the same as for the unit figures.

Constant practice in the use of these tables is essential.

Exercise XIV (d)

Using the tables, determine the logs of the following numbers:

1. 1·65, 16·5, 165, 1650, 0·165, 0·0165, 0·00165.

2. 3, 30, 300, 0·3, 328, 32·8, 0·00328.

3. 89·61, 8961, 0·8961, 5060, 5·006, π.

4. 1, 0·1, 0·001, 10·01, 100·2, 1002.

§ 14. Antilogarithms.

The booklet contains also a set of tables under the title "Antilogarithms." These give the numbers corresponding to given logarithms.

MATHEMATICAL TABLES.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3 4	5	6 7 8 9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1 1	1	1 2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 1 1 1	1	1 2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1 1	1	1 2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1 1	1	1 2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1 1	1	2 2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1 1	1	2 2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1 1	1	2 2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1 1	1	2 2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1 1	1	2 2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1 1	1	2 2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1 1	1	2 2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1 1	2	2 2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1 1	2	2 2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1 1	2	2 2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1 1	2	2 2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1 1	2	2 2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1 1	2	2 2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1 1	2	2 2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1 1	2	2 2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1 1	2	2 3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1 1	2	2 3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1 2	2	2 3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1 2	2	2 3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1 2	2	2 3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1 2	2	2 3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1 2	2	2 3 3 4

These tables are used in exactly the same way as the logarithm tables. Remember, however, that only the mantissa of the log is used in determining the digits which build up the number.

Thus:

(i) To find the antilog of 1·2474, that is to find the number of which the log is 1·2474.

In the antilog tables, find the number corresponding to the mantissa ·2474, viz. 1766 + 2, i.e. 1768. These are the digits composing the number, but the position of the decimal point depends upon the characteristic. In this case the characteristic is 1, and this we know to be one less than the number of figures before the decimal point. There are, therefore, 2 figures before the decimal point, and the number of which the log is 1·2474 is therefore 17·68.

(ii) To find antilog $\bar{3}$·0456.

Referring to the tables, the antilog of ·045 is 1109. For the next figure 6, add 2, and the antilog of ·0456 is found to be 1111

The characteristic $\bar{3}$, being negative, shows that the number is a decimal fraction, and that there are two noughts directly after the decimal point.

Hence, antilog $\bar{3}$·0456 is 0·001111.

It is in the placing of the decimal point that most mistakes are made. Before leaving your answer, check it by seeing that the characteristic of its log agrees with that given.

Exercise XIV (e)

From the tables, find the antilogs of:

1. 0·1684, 2·1684, $\bar{2}$·1684, $\bar{1}$·1684.
2. 1·5672, $\bar{1}$·5672, 3·5672.
3. Arrange − 0·7313 so that only the characteristic is negative, then determine its antilog.
4. Of what number is 0 the logarithm?
5. Arrange − 2·3642 so that only the characteristic is negative, then determine its antilog.
6. Determine by logs $(2{\cdot}63)^{-3}$.
7. Using logarithms, calculate $\dfrac{1}{6{\cdot}034}$ and $\dfrac{1}{38{\cdot}35}$.
8. Calculate $(326{\cdot}4)^{-\frac{1}{3}}$.

§ 15. Applications.

Remember that logarithms cannot be applied to sums and differences.

Example i.—Evaluate $\dfrac{83{\cdot}69 \times 2{\cdot}685 \times 0{\cdot}384}{97{\cdot}64 \times 0{\cdot}067}$.

The following is a convenient way to set out the computation. Write A for the answer, then

$$\log A = \overset{\text{Numerator.}}{(\log 83{\cdot}69 + \log 2{\cdot}685 + \log 0{\cdot}384)}$$

$$\overset{\text{Denominator.}}{-(\log 97{\cdot}64 + \log 0{\cdot}067)}$$

$$= \begin{array}{r} \left\{\begin{array}{r} 1{\cdot}9227 \\ 0{\cdot}4289 \\ \bar{1}{\cdot}5843 \end{array}\right\} \\ \hline 1{\cdot}9359 \\ -\,0{\cdot}8157 \\ \hline 1{\cdot}1202 \end{array} \qquad - \begin{array}{r} \left\{\begin{array}{r} 1{\cdot}9896 \\ \bar{2}{\cdot}8261 \end{array}\right\} \\ \hline 0{\cdot}8157 \end{array}$$

$$\therefore \quad A = \text{antilog } 1{\cdot}1202$$

$$= 13{\cdot}19.$$

Example ii.—Evaluate $\sqrt[3]{\dfrac{83{\cdot}69 \times 0{\cdot}2685 \times 0{\cdot}0384}{97{\cdot}64 \times 0{\cdot}67}}$.

$$\log A = \tfrac{1}{3}[(\log 83{\cdot}69 + \log 0{\cdot}2685 + \log 0{\cdot}0384)$$

$$-(\log 97{\cdot}64 + \log 0{\cdot}67)]$$

$$= \tfrac{1}{3}\left[\begin{array}{r} \left\{\begin{array}{r} 1{\cdot}9227 \\ \bar{1}{\cdot}4289 \\ \bar{2}{\cdot}5843 \end{array}\right\} \\ \hline \bar{1}{\cdot}9359 \\ -\,1{\cdot}8157 \\ \hline \bar{2}{\cdot}1202 \end{array} \qquad - \begin{array}{r} \left\{\begin{array}{r} 1{\cdot}9896 \\ \bar{1}{\cdot}8261 \end{array}\right\} \\ \hline 1{\cdot}8157 \end{array} \right]$$

$$= \frac{\bar{3} + 1{\cdot}1202}{3}$$

$$= \bar{1}{\cdot}3734;$$

$$\therefore \quad A = \text{antilog } \bar{1}{\cdot}3734$$

$$= 0{\cdot}2362.$$

Example iii.—Evaluate

$$\sqrt{\frac{6{\cdot}728 \times (2{\cdot}87)^3}{15{\cdot}34} + \frac{1{\cdot}374 \times \sqrt{46{\cdot}47}}{729{\cdot}23}}.$$

The two parts connected by the plus sign must be evaluated separately and the results added. Let the answers to the parts be denoted by A_1 and A_2 respectively.

$$\log A_1 = \tfrac{1}{2}[\log 6\cdot728 + 3\log 2\cdot87 - \log 15\cdot34]$$

$$= \tfrac{1}{2}\left[\begin{array}{l} 0\cdot8279 + 3 \times 0\cdot4579 - 1\cdot1858 \\ +1\cdot3737 \\ \hline 2\cdot2016 \\ -1\cdot1858 \\ \hline 1\cdot0158 \end{array}\right]$$

$$= 0\cdot5079;$$

$$\therefore\ A_1 = \text{antilog } 0\cdot5079$$
$$= 3\cdot221.$$

$$\log A_2 = \log 1\cdot374 + \tfrac{1}{2}\log 46\cdot47 - \log 729\cdot2$$

$$= \left\{\begin{array}{l} 0\cdot1379 + \dfrac{1\cdot6672}{2} - 2\cdot8628 \\ +0\cdot8336 \\ \hline 0\cdot9715 \\ -2\cdot8628 \end{array}\right.$$

$$= 2\cdot1087$$

$$\therefore\ A_2 = \text{antilog } \bar{2}\cdot1087$$
$$= 0\cdot01284,$$

$$A_1 + A_2 = \begin{array}{r} 3\cdot221 \\ +0\cdot01284 \\ \hline 3\cdot23384 \end{array}$$

Exercise XIV (f)

Using logarithms, compute:

1. $\dfrac{3\cdot28 \times 15\cdot23}{8\cdot42}$. 2. $\dfrac{\pi(6\cdot25)^2}{2\cdot85}$. 3. $\dfrac{263 \times 10\cdot8 \times 496}{4\cdot198\pi}$.

4. $\sqrt[3]{3\cdot082}$. 5. $15(6\cdot32)^{1\cdot34}$. 6. $\pi\sqrt{\dfrac{126\cdot3 \times 2\cdot005}{3\cdot28 \times 54\cdot8}}$.

7. $(6\cdot345 \times 0\cdot1075)^{2\cdot5} \div (0\cdot00374 \times 96\cdot37)^3$.

8. (i) $\sqrt[3]{20760}$. (ii) $\sqrt[3]{0\cdot02076}$. 9. $\dfrac{3\cdot024 \times \sqrt{0\cdot1275} \times 73\cdot24}{\sqrt[3]{2\cdot124} \times 32\cdot78}$.

10. Solve the equation $5^x = 120$.

11. (i) $(1{\cdot}03)^{50}$. (ii) $(\frac{2}{19})^{\frac{1}{5}}$. (iii) $(1{\cdot}04)^{7}$.

12. $(22{\cdot}15 \div 4{\cdot}139)^{0{\cdot}86}$.

13. $(55{\cdot}21)^2 \times 3{\cdot}142 \div 2{\cdot}206$.

14. $\left(\dfrac{28{\cdot}68 \times 0{\cdot}0173}{0{\cdot}00197}\right)^{0{\cdot}4}$.

15. $\sqrt[5]{\dfrac{\tan 40^\circ}{65}}$.

16. Find x when $2^{x+1} = 3^{x-1}$.

17. $\dfrac{3{\cdot}862 \times \sqrt{13{\cdot}25}}{11{\cdot}28} + 3{\cdot}52$.

18. $\dfrac{\pi(3{\cdot}85)^2}{0{\cdot}58} - \dfrac{\sqrt{6{\cdot}54}}{0{\cdot}76}$.

19. $12{\cdot}68\sqrt{0{\cdot}057} + \log \tan 55^\circ$.

20. $\log \dfrac{\sin 60^\circ}{\cos 45^\circ} + \log(\sin 60^\circ \cos 45^\circ)$.

§ 16. A Simple Slide Rule.

Take two strips of cardboard, and on an edge of each mark off lengths corresponding to the logarithms of the numbers from 1 to 10, and then of the tens from 10 to 100.

This can be done from the graph on page 145 by placing the strips along the ordinates, and marking off their lengths.

The lengths for the tens can be obtained at once from those of the units; for, say, $\log 50 = \log 10 + \log 5$, i.e. the length for 50 is the sum of the lengths for 10 and 5.

The scale constructed is called a logarithmic scale. The lengths upon it represent the logarithms of the numbers, and are not proportional to the actual numbers.

See that the scales are marked as shown in the figure.

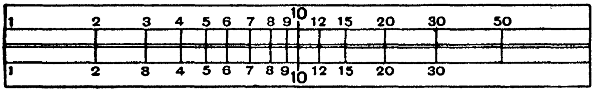


Fig. 2.

To check the rule, find the products of simple numbers, say 2 and 3, in the following manner:

Move the lower scale to the right until its mark 1 is opposite 2 on the upper scale. Then the reading on the upper scale, which is opposite the 3 on the lower scale, should be the product of 2 and 3. Notice that in this operation the logarithms of 2 and 3 have been added.

Division is the reverse operation. To divide 6 by 3, place the lower scale so that the 3 mark is opposite the 6 mark on the upper scale. Then the mark on the upper scale to which the 1 on the lower scale is opposite is the quotient.

Now mark on the scales, lengths corresponding to 12, 15, 16, 18, 25, 35, 45, etc.

In order to test your rule, find by its means:

$$\text{(i) } 5\times 3, \quad \text{(ii) } 5^2, \quad \text{(iii) } 8\times 5, \quad \text{(iv) } \cdot 5\times 3 \left(\text{i.e. } \frac{5\times 3}{10}\right).$$

CHAPTER XV

THE QUADRATIC GRAPH

§ 1. Graphs of Expressions containing the second, but no higher power.

Take the simplest expression, x^2, and plot its values for different values of x.

-4	-3	-2	-1	$=x=$	0	1	2	3	4
16	9	4	1	$=x^2=$	0	1	4	9	16

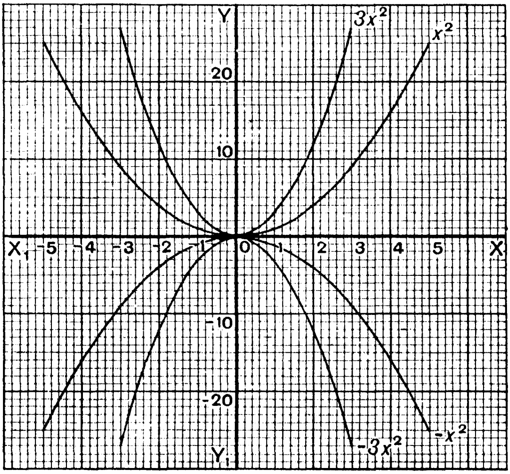


Fig. 1.

Examine the graph, and verify the following statements:

(i) The graph is not a straight line, but a curve with a vertex and two diverging branches.

(ii) The values of x^2 are all positive.

(iii) The graph touches the axis of x, the vertex, in this case, being at the origin.

(iv) The graph is symmetrical about the axis of y; i.e. if the paper is folded so that the crease is along the axis of y, one branch of the curve will coincide with the other branch.

(v) The curve grows straighter away from the vertex.

(vi) The gradient changes.

(vii) The gradient is positive (up) on the right, zero at the vertex, and negative (down) on the left.

§ **2.** On the same axes, plot the graph of $3x^2$, and compare it with that of x^2.

Observe: (i) The gradient of this graph changes more rapidly.

(ii) The coefficient 3 appears as the value of $3x^2$ when x equals 1 or -1.

§ **3.** On the same axes, plot the graphs of $-x^2$ and $-3x^2$, and compare them with those of x^2 and $3x^2$.

You will conclude that the inversion is due to the change in sign.

§ **4.** To determine the effect of an added constant, plot the graphs of:

(i) $y = 3x^2 + 2$. (ii) $y = 3x^2 - 2$.

(iii) $y = -3x^2 + 2$. (iv) $y = -3x^2 - 2$.

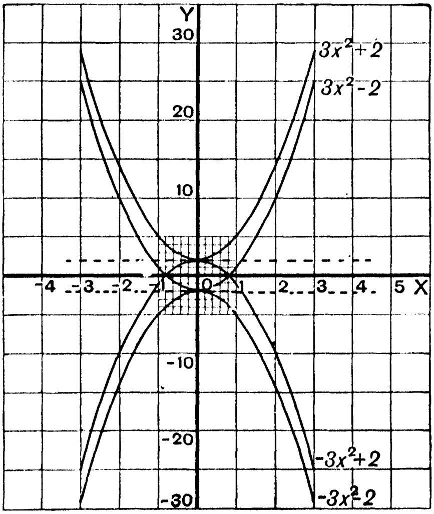


FIG. 2.

Comparing these with the other graphs, it is seen that:

(i) The effect of the 2 has been to raise the graph $3x^2$ through a distance 2 above the origin.

Observe that the graph has the same shape, and is still symmetrical about the axis of y, also that the minimum value of $3x^2+2$ is 2, this being shown by the position of the vertex.

If a new axis of x is drawn through the point $y=2$, then the coefficient 3 appears as the new value of y when x is 1.

(ii) The effect of the -2 has been to lower the graph of $3x^2$ through a distance 2 below the origin.

The graph of $3x^2-2$ cuts the axis of x at two points, and at these the value of $3x^2-2$ is 0. The intersections of the axis of x give the corresponding values of x. Observe that one is positive and the other negative.

These values of x are called the roots of the equation,

$$3x^2-2=0,$$

from which

$$3x^2=2,$$

$$x^2=\tfrac{2}{3},$$

$$x=\pm\sqrt{\tfrac{2}{3}}.$$

The graph of $3x^2+2$ shows that the equation $3x^2+2=0$ has no roots, because the graph does not intersect the axis of x. In such cases the roots are said to be imaginary.

(iii) The graph of $-3x^2+2$ shows that there are two roots to the equation

$$-3x^2+2=0.$$

(iv) The graph of $-3x^2-2$ shows that the roots of the equation

$$-3x^2-2=0$$

are imaginary.

Summary.

1. The graph of an equation of the form $ax^2\pm c$ is a curve. It is called a **Parabola.**

2. The graph is symmetrical about the axis of Y.

3. If the coefficient of x is positive, the vertex is downwards; if negative, upwards.

4. When the added constant is positive, the vertex is above the origin; when negative, below the origin.

§ **5.** To straighten the graph of ax^2+c.

If the values of $3x^2$ are plotted against the values of x^2 instead of x, the graph shown in the figure is obtained.

-3	-2	-1	$=x=$	0	1	2	3	4
9	4	1	$=x^2=$	0	1	4	9	16
27	12	3	$=3x^2=$	0	3	12	27	48

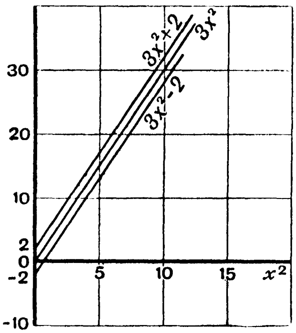


FIG. 3.

The graph is seen to be a straight line of gradient 3.

Similarly, plot the values of

(i) $3x^2+2$, (ii) $3x^2-2$, (iii) $-3x^2+2$, (iv) $-3x^2-2$,

against x^2.

Examine the graphs, and verify that:

(i) All the graphs are straight lines.

(ii) The gradient in each case is the coefficient of x^2.

(iii) The added constant is shown on the axis of Y.

That the graph should be a straight line will be readily understood if, say, z is substituted for x^2.

Then, $3x^2=3z$, and the equation of the graph becomes $y=3z$.

This result is very useful in testing whether a given set of numbers follow an assumed law.

Suppose that when certain values depending upon x are plotted against x, a graph is obtained which looks like those of expressions containing x^2; then, in order to test your supposition, it is only necessary to plot the values against x^2 instead of x.

If the resulting graph is a straight line, your assumption is correct.

Moreover, the expression can be written down at once, for the gradient gives the coefficient of x^2, and the intersection of the axis of Y the added constant.

Application.

A good example is found in establishing the law of the pendulum.

The following numbers, showing the time of swing of simple pendulums of different lengths, were obtained by a class of boys aged 13 to 14 years :

Time (secs.), t	1·05	1·6	1·76	2	2·45
Length (cms.), l	25	50	75	100	150

On plotting l against t, the graph obtained appeared to be one of the branches of a parabola. When l was plotted against t^2 the graph was found to be approximately a straight line, the gradient of which was 25 and the added constant 0. Verify this.

The relation is therefore $l = 25t^2$, or $t = 0{\cdot}2\sqrt{l}$.

This agrees closely with the formula

$$t = 2\pi\sqrt{\frac{l}{g}}, \text{ where } g = 981.$$

§ **6.** In order to determine the effect of introducing the first power of x into the expression, plot the graph of $x^2 + x$.

-3	-2	-1	$= x =$	0	1	2	3
6	2	0	$=(x^2+x)=$	0	2	6	12

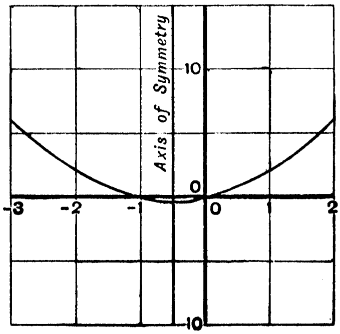


FIG. 4.

Comparing this graph with that of x^2, it is seen that the effect of x is to displace, as it were, the graph of x^2 towards the second quadrant, and to lower it.

The graph cuts the axis of X at two points, and the axis of symmetry is no longer the axis of Y, but an axis mid-way between the two values of x for which $x^2+x=0$, i.e. at $x=-\frac{1}{2}$.

Observe also that for this value of x the value of x^2+x is a minimum.

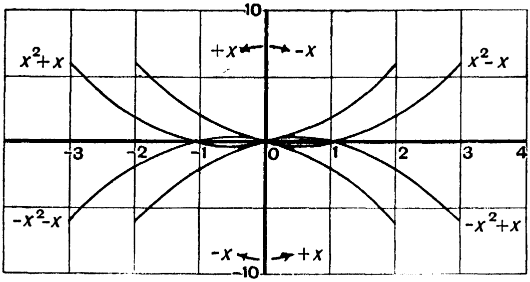


FIG. 5.

The axis of symmetry is readily found as follows:

$$x^2+x=\{x^2+x+(\tfrac{1}{2})^2\}-(\tfrac{1}{2})^2=(x+\tfrac{1}{2})^2-\tfrac{1}{4}.$$

If z is written for $(x+\frac{1}{2})$, the expression becomes $z^2-\frac{1}{4}$.

If z is measured along the axis of x, but from the point $x=-\frac{1}{2}$, and if the expression $z^2-\frac{1}{4}$ is measured on a vertical axis through this point, the graph of $z^2-\frac{1}{4}$ is symmetrical about this vertical axis, and cuts it at the point $-\frac{1}{4}$.

Thus the axis of symmetry passes through the point $x=-\frac{1}{2}$.

Plot and examine the graphs of:

(i) x^2-x. (ii) $-x^2+x$. (iii) $-x^2-x$.

Verify that the vertices of the graphs of x^2 and $-x^2$ are displaced in the counter-clockwise direction by $+x$, and in the clockwise direction by $-x$ (fig. 5).

§ **7.** We are now able to examine the graph of an expression containing x^2, x and a constant.

Plot the graph of, say, $2x^2-4x-3$:

-2	-1	$=x=$	0	1	2	3	4
13	3	$=(2x^2-4x-3)=$	-3	-5	-3	3	13

The axis of symmetry evidently passes mid-way between $x=0$ and $x=2$, i.e. through $x=1$.

Draw the axis, make it a new axis of Y, and reckon values along the axis of X from this new axis of Y.

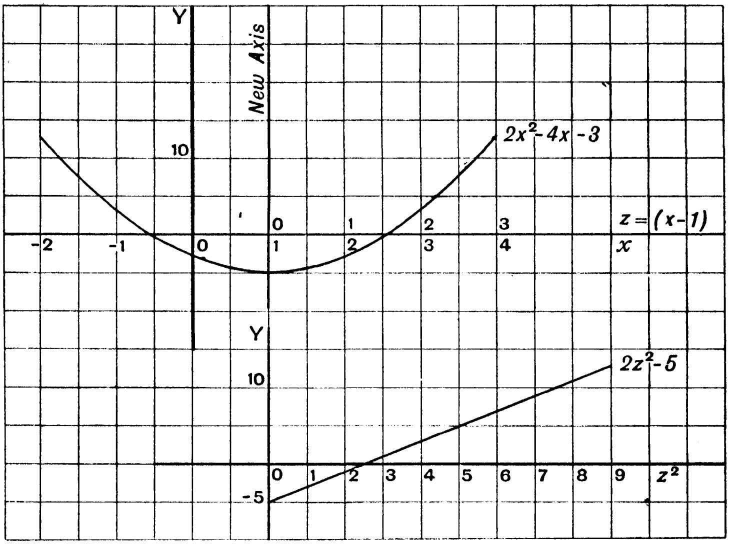


FIG. 6.

Call the new values of x, z; then

$$z=(x-1) \quad \text{and} \quad x=(z+1).$$

Substituting this value of x in the given expression, we have

$$\begin{aligned}2x^2-4x-3 &= 2(z+1)^2-4(z+1)-3\\ &= 2z^2-5.\end{aligned}$$

This expression contains only the second power of z.

Notice that -5 is the minimum value of the original expression.

If, therefore, values of $2z^2-5$, i.e. of $2x^2-4x-3$, are plotted against z^2, the graph is a straight line of gradient 2, which intersects the new axis of Y at -5 (fig. 6).

The original equation can be obtained from the equation to this straight line by substituting $(x-1)$ for z.

The importance of this is, that we can test whether a given graph is of the form $ax^2 + bx + c$, a, b and c being constants.

The method is as follows:

(i) Draw the axis of symmetry.

(ii) Reckon values of x from the axis of symmetry.

(iii) Plot the values of y against the square of the new values of x.

If the graph (iii) is a straight line, the assumption is correct. From the equation to graph (iii) the expression can be found.

§ **8.** A quadratic expression, i.e. an expression of the form $ax^2 + bx + c$, can be determined if its values for three known values of x are known.

Example.—The values of a quadratic expression are 6, 2 and 3, when the values of x are 1, -1 and 2 respectively. Find the expression. Substitute the known values in the general equation:

$$ax^2 + bx + c = y;$$

then
$$a + b + c = 6, \text{ when } x = 1.$$
$$a - b + c = 2, \text{ when } x = -1.$$
$$4a - 2b + c = 3, \text{ when } x = -2.$$

Solve these simultaneous equations for a, b and c.

It will be found that a is 1, b is 2 and c is 3.

The expression is therefore $x^2 + 2x + 3$.

Exercise XV

Find the quadratic expressions which have the given values for the stated values of x:

1. -5, when x is 1; 40, when x is -4; -2, when x is 2.
2. -3, when x is 0; 6, when x is 3; -19, when x is -2.

Find the functions of x which have the following values for the given values of x:

3. 15, when x is -2; 10, when x is 3; 1, when x is 0.
4. -5, when x is 1; -5, when x is -1; -6, when x is 0.
5. Draw a graph of $x = y^2$.
6. If $x^2 + 2x + 3$ is a function of x, express it as a function of z, where $z = (x + 1)$.

7. Plot the graph of $2x^2$, and on the same axes, the graph of $5x - 3$.

Add the ordinates of the two graphs, and compare the graph obtained with that of $2x^2 + 5x - 3$.

8. The following are corresponding values of x and y. Find the equation connecting them:

x	-3	-2	-1	0	1	2	3
y	6	3	2	3	6	11	18

9. Trace the graphs of $y = 2x^2$ and $y = 5x + 3$, and from them find the roots of the equation $2x^2 - 5x = 3$.

10. Draw a graph of $y = x(a - x)$ from $x = 0$ to a. From the graph, show that if the sum of two positive numbers is given, their product is increased by making the numbers more nearly equal.

11. The following numbers show the weight of circular sheet iron discs of different diameters. Plot the numbers, and find the law connecting weight and diameter.

Diameter (cms.)	1	2	4	6	8	10
Weight (grms.)	·785	3·14	12·57	28·27	50·28	78·55

12. The distance through which a body falls under gravity is given by the following table:

Time (secs.)	0	1	2	3	4	5
Distance (ft.)	0	16	64	144	256	400

Measure distance on the vertical axis, and time on the horizontal axis, and construct a graph showing the relation between distance and time.

The velocity acquired by the body is given in the following table:

Time (secs.) -	0	1	2	3	4	5
Velocity (ft. p. sec.)	0	32	64	96	128	160

Represent the velocity on the other side of the axis used for distance in the distance-time graph, and using the same axis of time, plot on the same figure the graph of velocity and time.

By means of these graphs, find the velocity acquired by the body when it has fallen through (i) 36 ft., (ii) 100 ft., (iii) 200 ft., (iv) 320 ft.; and the distance through which it has fallen when its velocity is (i) 40 ft. per sec., (ii) 80 ft. per sec., (iii) 100 ft. per sec., (iv) 136 ft. per sec.

13. Show by Algebra, in the manner used for x^2+x, that the axis of symmetry of the graph x^2-4x-3 passes through $x=2$, and that the minimum value is therefore -7.

14. Express $2x^2+4x-3$ in the form $2z^2 \pm$ a number. What is the value of z? Then show that the axis of symmetry passes through $x=-1$, and that the minimum value of $2x^2+4x-3$ is therefore -5.

15. The value of the bending moment (M) at a distance x from the centre of a beam of length l, supported at each end and loaded uniformly at w lbs. per foot, is given by the equation

$$M = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right).$$

What do you know concerning the graph obtained by plotting M and x?

For what value of x will M be a maximum?

Choosing suitable values for w and l, construct the graph, taking l as the axis of x.

16. When a beam fixed at one end only is uniformly loaded at w lbs. per unit length, the bending moment (M) at a distance x from the free end is given by the equation

$$M = -\frac{wx^2}{2}.$$

Construct a graph connecting M and x, and draw as many conclusions from it as you can.

17. Why is the axis of symmetry of the graph of ax^2+bx+c the same as that of the graph of ax^2+bx?

18. Draw the graphs of $y=x^2-4$ and $y=4-x^2$. What do the points of intersection indicate?

19. The following tables show the solubility of nitre and potassium chlorate, in grams per 100 grams of water, at various temperatures. See if in each case the numbers follow approximately the law, $y = ax^2 + bx + c$, and if so, find suitable values of a, b and c.

Nitre.

Temp. ($x°$ C.) -	0	10	20	40	60
Solubility (y) -	13	21	31	63	110

Potassium Chlorate.

Temp. ($x°$ C.) -	0	30	50	80	90
Solubility (y) -	2·5	9	18	40	49

CHAPTER XVI

QUADRATIC EQUATIONS

A QUADRATIC equation contains the second but no higher power of the unknown. It may or may not contain the first power.

§ 1. Pure Quadratic.

A pure quadratic equation contains only the second power of the unknown.

EXAMPLE. $3x^2 = 108.$

To solve this equation, find x^2, and then extract the square root.

Thus:

$$\begin{aligned} 3x^2 &= 108, \\ x^2 &= 36, \\ x &= \pm\sqrt{36} \\ &= \pm 6. \end{aligned}$$

Substituting these values for x, it will be found that both $+6$ and -6 satisfy the equation.

§ 2. Adfected Quadratic.

An adfected quadratic equation contains both the second and the first power of the unknown.

EXAMPLE. $x^2+2x-8=0$ or $x^2+2x=8$.

It has been seen already, that such equations can be solved graphically, or by the method of factors (page 129).

There is another method, which is more useful when the factors are not at once apparent.

It consists of making sure that the side of the equation containing the unknown is a perfect square; then, on taking the square root, the equation is reduced to one containing the first power of the unknown only.

EXAMPLE.—Solve the equation $2x^2-7x-22=0$.

i. Take the number -22 to the other side, leaving on the left the terms containing x.

ii. Reduce the coefficient of x^2 to 1, by dividing both sides by 2.

iii. Complete the square on the left, by adding the third term, viz. $(-\frac{7}{4})^2$. Add the same number to the other side.

iv. Extract the square root of both sides.

$$2x^2-7x=22,$$
$$x^2-\tfrac{7}{2}x=11,$$
$$x^2-\tfrac{7}{2}x+(-\tfrac{7}{4})^2=11+\tfrac{49}{16},$$
$$(x-\tfrac{7}{4})^2=\tfrac{225}{16},$$
$$(x-\tfrac{7}{4})=\pm\sqrt{\tfrac{225}{16}}$$
$$=\pm\tfrac{15}{4},$$
$$x=\tfrac{7}{4}\pm\tfrac{15}{4},$$

i.e. $x=\frac{7}{4}+\frac{15}{4}=5\frac{1}{2}$ or $x=\frac{7}{4}-\frac{15}{4}=-2$.

Substituting these values of x in the given equation, it will be found that the equation is satisfied by either value of x.

This agrees with the graphical illustration on page 165.

EXERCISE XVI (A)

Solve the equations. (*Check your answers by substitution.*)

1. $(x-1)^2=4$.
2. $3(x+1)^2=27$.
3. $(x+1)^2=1$.
4. $(x-3)^2=0$.
5. $2x^2-16x+32=0$. What do you notice concerning the roots? Account for it if you can.
6. $x^2+21x+120=10$.
7. $6x^2+13x=-6$.
8. $6x^2-1=5x$.
9. $x(x-2)+4x=3$.
10. $6x^2-17x+12=0$.
11. $10x^2+11x-35=0$.
12. $4x^2+3x-5=2$.
13. $(4x-1)(2x+3)-(10x+3)(5-x)=2(3x-7)^2+25-94x^2$.

In 14 to 24, clear of fractions first.

14. $\dfrac{1}{x+2}+\dfrac{2}{2x+1}+\dfrac{4}{2x-1}=0.$

15. $\dfrac{x}{2}+\dfrac{x-1}{3}+\dfrac{x-2}{4x}=1.$

16. $\dfrac{1}{x-2}-\dfrac{1}{x+4}=\dfrac{1}{x}.$

17. $\dfrac{x}{x+3}+\dfrac{x+3}{x+5}=0.$

18. $\dfrac{x+3}{x-3}-\dfrac{x-3}{x+3}=6\tfrac{6}{7}.$

19. $\dfrac{3x}{x+1}=6-\dfrac{4x+1}{2x+1}.$

20. $\dfrac{7x}{x+2}=\dfrac{5x+1}{3x+1}.$

21. $\dfrac{2-3x}{2+3x}-\dfrac{2+3x}{2-3x}=2.$

22. $\dfrac{3x+5}{3x-1}-\dfrac{5x^2}{9x^2-1}=\dfrac{3x+8}{3x+1}.$

23. $\dfrac{1}{2(x-1)}+\dfrac{3}{x^2-1}=\dfrac{1}{4}.$

24. $\dfrac{5x-7}{10x-5}=\dfrac{1}{10}-\dfrac{4x-3}{4x-2}.$

25. $\dfrac{2}{x^2}+\dfrac{5}{x}=18.$

§ 3. The General Quadratic Equation.

In the general quadratic equation, the coefficients and the number are represented by letters, say a, b and c.

The general equation is, then,

$$ax^2+bx+c=0.$$

This can be solved by the method of completing the square, thus:

$$ax^2+bx=-c,$$

$$x^2+\frac{bx}{a}=-\frac{c}{a}.$$

Complete the square by adding $\left(\dfrac{b}{2a}\right)^2$ to each side:

$$x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a},$$

$$\text{i.e.}\quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

$$\therefore\quad x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}.$$

That is, $x=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ or $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

If this result be remembered, the roots of quadratic equations can be written down immediately.

Thus: Write down the roots of $12x^2-5x-2=0$.

Here $\qquad a = 12, \quad b = -5 \quad \text{and} \quad c = -2.$

$$\therefore\ x = \frac{-(-5) + \sqrt{(-5)^2 - 4 \times 12 \times -2}}{2 \times 12}$$

$$\text{or} \quad \frac{-(-5) - \sqrt{(-5)^2 - 4 \times 12 \times -2}}{2 \times 12}$$

$$= \frac{5 + \sqrt{25 + 96}}{24} \quad \text{or} \quad \frac{5 - \sqrt{25 + 96}}{24}$$

$$= \frac{2}{3} \qquad \text{or} \quad -\frac{1}{4}.$$

Return to Exercise XII (D), and write down the roots from the roots of the general equation, by the method just given.

The Discriminant.

Returning to the consideration of the roots of the general quadratic equation $ax^2 + bx + c = 0$,

$$\text{viz.} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The part $b^2 - 4ac$ is so important that it is called the **Discriminant.** The nature of the roots depends upon the discriminant.

For example:

(1) If $b^2 - 4ac$ (the discriminant) $= 0$, the roots of the equation are each equal to $-\frac{b}{2a}$.

(2) If $\sqrt{b^2 - 4ac}$ is numerically greater than b, then one root is positive and the other negative.

(3) If $\sqrt{b^2 - 4ac}$ is numerically less than b, then both roots are negative.

(4) If $\sqrt{b^2 - 4ac} = b$, then one root is zero.

(5) If $4ac$ is greater than b^2, the discriminant is negative, and since there is no real square root of a negative number, the roots are said to be imaginary.

These results and others are illustrated graphically in the next chapter.

§ 4. Applications.

1. A chord at right angles to the diameter of a circle.

Such a chord is bisected by the diameter.

Let the radius of the circle be R, the length of the semi-chord, c, and the shorter portion of the diameter cut off by the chord, t.

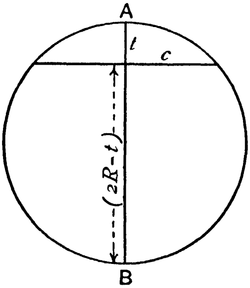


FIG. 1.

The other portion of the diameter is, then, $(2R - t)$.

By an important theorem in Geometry,

$$c^2 = (2R - t)t,$$

from which $$c = \pm\sqrt{(2R - t)t}.$$

The whole chord is, therefore, $\pm 2\sqrt{(2R - t)t}$.

EXAMPLE.—In a circle of 5 cms. radius, a chord of length 6 cms. is drawn at right angles to a diameter. Find where the chord intersects the diameter.

From $$c^2 = (2R - t)t,$$

$$. \quad t^2 - 2Rt = -c^2.$$

Solving this quadratic,

$$t^2 - 2Rt + (-R)^2 = R^2 - c^2,$$

$$t - R = \pm\sqrt{R^2 - c^2},$$

$$t = R \pm \sqrt{R^2 - c^2}.$$

From the example, R is 5 and c is 3.

Therefore $$t = 5 \pm \sqrt{25 - 9}$$

$$= 5 \pm 4 = 9 \text{ or } 1.$$

You will observe that the roots are the distances from both ends of the diameter, or regarded from one end, say A, the chord may be 1 cm. or 9 cms. from that end.

2. To divide a straight line into two parts, such that the square on one part shall be equal to the rectangle contained by the whole line and the other part.

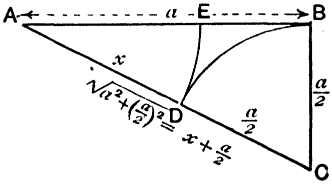


Fig. 2.

Let the length of the given straight line AB be a units, and that of one of the required parts x units. It follows that the length of the other part is $(a-x)$ units.

From the conditions of the problem,

$$x^2 = a(a-x),$$

from which $$x^2 + ax = a^2.$$

Complete the square, $x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \left(\frac{a}{2}\right)^2$.

Find $a^2 + \left(\frac{a}{2}\right)^2$ geometrically.

Extract the square-root, $x + \frac{a}{2} = \pm\sqrt{a^2 + \left(\frac{a}{2}\right)^2}$ (AC in fig. 2).

Taking the plus sign,
$$\begin{aligned} x &= \text{AC} - \frac{a}{2} \\ &= \text{AC} - \text{CB} \\ &= \text{AD}. \end{aligned}$$

From A, mark off along AB, AE equal to AD; then the square on AE is equal to the rectangle AB × BE.

Exercise.—Take the minus sign of AC, and obtain

$$x = -\text{AC} - \frac{a}{2}.$$

Complete the construction and interpret the result.

3. **Trigonometrical Ratios of 36°, etc.**

Referring to the figure, having divided AB at E as shown in Application 2, carry out the following construction:

Bisect EB at F; draw FK $\perp^r$ to EB; with centre A and radius AB, cut FK at G.

Join AG, GB and GE.

Then AGB, GEB and AEG are isosceles triangles with AB = AG, GB = GE and AE = EG respectively.

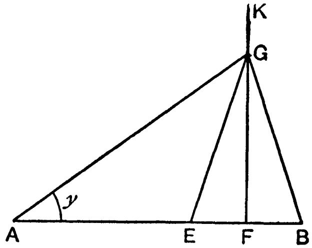


FIG. 3.

Let $\angle \text{BAG} = y°$;

then $\angle \text{EGA} = y°$

and extr. $\angle \text{GEB} = 2y°$.

But $\angle$GEB $= \angle$EBG or $\angle$ABG, and $\angle$ABG $= \angle$AGB.

Hence $\angle$ABG and $\angle$AGB each equal $2y°$, i.e. ABG is an isosceles triangle having its equal angles each double the remaining angle.

Knowing that the three angles of a triangle equal 180°, calculate the value of each angle; and from the facts that

$$\text{AE} = \frac{\sqrt{5a^2}}{2} - \frac{a}{2} \quad \text{or} \quad \frac{a}{2}\left(\sqrt{5} - 1\right),$$

and that AB and AG each equal a, find AF and GF. Thence find the trigonometrical ratios of 36,° 18,° 54° and 72.°

Compare with the values given in tables.

4. The Length of Tangents

PT is a tangent drawn to a circle of radius R, from a point P, at a distance d from the centre O (fig. 4).

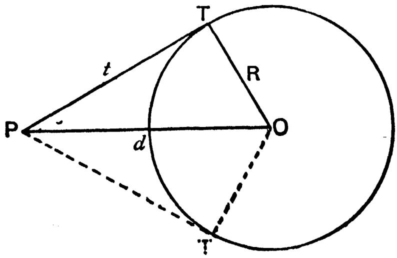


FIG. 4.

The angle PTO is a right angle.

Hence
$$PT^2 = OP^2 - OT^2$$
$$= d^2 - R^2.$$

If the length of the tangent is t,
$$t^2 = d^2 - R^2$$
or
$$t = \pm\sqrt{d^2 - R^2}.$$

Notice that $\cos\angle TOP = \frac{R}{d}$. This enables you to find the angle and the minor or major arc TT.

5. **The Horizon.**

To an observer at P (fig. 5), T is a point on the horizon which is a circle having its centre on PO (the straight line from P to the centre of the earth).

In calculations upon the horizon, the distance (d) of the observer from the centre of the earth is very little greater than the radius (R) of the earth.

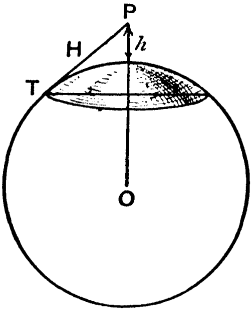


FIG. 5.

If H is the distance of the horizon from P,
$$H = \sqrt{d^2 - R^2}$$
$$= \sqrt{(d + R)(d - R)}.$$

Now d is practically equal to R, and therefore $d + R$ to 2R, and $d - R$ is the height of P above the surface of the earth.

Call this height h; then
$$H = \sqrt{2Rh}.$$

If R and h are in miles, H will be in miles. It is usual to be given h in feet, in which case $\frac{h}{5280}$ must be placed in the formula instead of h. Substituting 4000 miles for R, the formula becomes:

$$H_{(\text{miles})} = \sqrt{\frac{8000}{5280}h^{(\text{ft.})}}$$

$$= 1\cdot231\sqrt{h^{(\text{ft.})}}.$$

EXAMPLE.—Find the distance of the horizon from an observer in a "crow's nest," 100 feet above the surface of the sea.

$$H = 1\cdot231\sqrt{100}$$

$$= 12\cdot31 \text{ miles.}$$

EXERCISE XVI (B)

1. The radius of a circle is 6 cms. Find the length of the chord at right angles to a diameter which it intersects 2 cms. from one end.
2. A chord measuring 4 cms. is $1\frac{1}{2}$ cms. from one end of the diameter of a circle which it intersects at right angles.
3. In Exercises 1 and 2, find the angle each chord subtends at the centre of the circle; also the area of the segments into which each chord divides the circle.
4. Divide a straight line 6 inches long into two parts which would contain a rectangle of area 7 sq. inches.
5. The cross-sectional area of a stream divided by the perimeter of the wetted part of the channel in which it flows is called the "Hydraulic Mean Depth" of the stream. Find the hydraulic mean depth when water to a depth of 6 inches flows through a pipe of diameter 10 inches.
6. The following table showing the distance of the visible horizon for various heights above sea level is taken from *Whitaker's Almanack*:

Height (feet) - - -	5	10	20	50	100	500	1000	5280
Distance of horizon (miles)	$2\frac{1}{4}$	4	$5\frac{3}{4}$	$9\frac{1}{4}$	$13\frac{1}{4}$	$29\frac{1}{2}$	$32\frac{1}{2}$	$95\frac{1}{2}$

Check these numbers by calculation.

7. In *Robinson Crusoe*, it is stated that Crusoe thought he saw the Peak of Teneriffe from his island. Taking his height above sea level to be 1000 feet and the Peak to be 12,200 feet high, calculate approximately how far he was from Teneriffe, assuming his conjectures to be correct.

8. Show how to draw an isosceles triangle, such that each angle at the base is twice the angle at the vertex.

   Let ABC and DEF be two such triangles, in which AB and AC are equal, and DE and DF are equal. If EF be equal to AB, show that the perimeter of the one triangle is to that of the other as 2 is to $\sqrt{5}+1$.

9. Determine the horizon for an observer in a Zeppelin at a height of 10,000 feet.

CHAPTER XVII

THE PROPERTIES OF QUADRATIC EXPRESSIONS AND EQUATIONS, SIMULTANEOUS EQUATIONS, PROBLEMS

§1. Quadratic Equations, Quadratic Form, Simultaneous Quadratics.

You have solved quadratic equations; that is, you have found the value of x for which a quadratic expression is equal to 0. Taking the equation $y=ax^2+bx+c$, when $y=0$,

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

These roots may take various forms. For example:

(i) They may both be positive, and different.

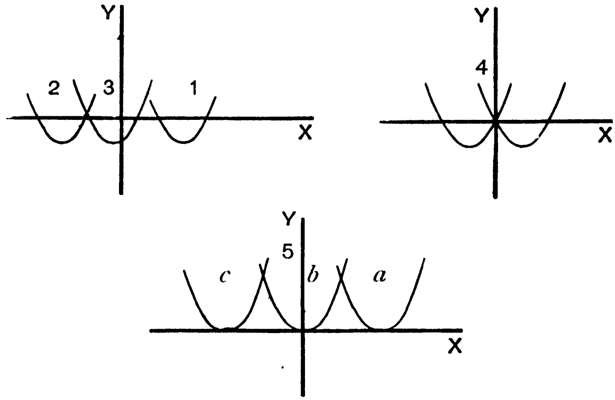
(ii) They may both be negative, and different.

(iii) One may be positive, and the other negative.

(iv) One may be zero, and the other positive or negative.

(v) They may be equal, and be positive, negative or zero.

These forms are illustrated in figs. 1 to 5.



FIGS. 1 TO 5.

§2. Referring to fig. 6, if A and B are the points at which the graph cuts the axis of X, then

$$AB = \text{the difference between the roots}$$
$$= \frac{\sqrt{b^2 - 4ac}}{a}.$$

If C is the point at which the graph cuts the axis of Y, and if CD is a straight line drawn across the parabola, parallel to the axis of X, then

$$CD = \text{the sum of the roots}$$
$$= \frac{-b}{a}.$$

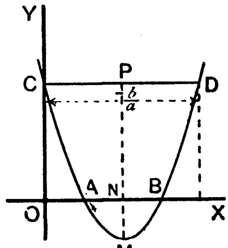


FIG. 6.

For, at point C, x is 0, and OC represents the value of $ax^2 + bx + c$ when x is 0, i.e. c.

There is, however, another point, namely D, at which the expression is equal to c. To find its co-ordinates, solve the equation

$$ax^2 + bx + c = c, \qquad ax^2 + bx = 0,$$
$$x(ax + b) = 0, \qquad x = 0 \text{ or } \frac{-b}{a}.$$

That is, the co-ordinates of the point D are $x = \frac{-b}{a}$ and $y = c$.

But CD is equal to the x co-ordinate of D, and is therefore equal to $\frac{-b}{a}$.

Hence CD is the sum of the roots.

The position and the direction of CD depend upon the signs of b, c and a. Draw the graphs and the inverse of these graphs, and illustrate this:

(i) $y = 2x^2 - 4x - 6$. (ii) $y = 2x^2 + 4x - 6$.

(iii) $y = 8x^2 + 16x + 6$. (iv) $y = 8x^2 - 16x + 6$.

Even when the roots are imaginary, CD represents their sum.

§ 3. Maximum and Minimum Values.

The vertex of the parabola gives the maximum or the minimum value of the expression.

When the vertex is upwards, the expression has a maximum value; when downwards, a minimum value.

Since the coefficient of x^2 is negative for the former, and positive for the latter, expressions containing $-x^2$ have a maximum value, and those containing $+x^2$ a minimum value.

Also, since the vertex is on the line bisecting CD at right angles (fig. 6), it follows that the value of x for the maximum or the minimum value of the quadratic expression is $\frac{CD}{2}$, i.e. $\frac{-b}{2a}$.

When x has this value, the expression $ax^2 + bx + c$ becomes:

$$\frac{-b^2}{4a} + c, \quad \text{i.e.} \quad \frac{4ac - b^2}{4a}.$$

EXAMPLE i.—Take the expression $(x^2 - 6x + 4)$.

In this case, $$\frac{-b}{2a} = \frac{-(-6)}{2} = 3.$$

When x is 3, the value of the expression is -5, and this is its minimum value.

Check the result by finding the value of the expression when x is 2, and when x is 4. It will be found that the value when x is 3 is less algebraically than when x is 2 or 4.

EXAMPLE ii.—Take the expression $(4 + 6x - x^2)$.

Here $$\frac{-b}{2a} = \frac{-6}{-2} = 3.$$

The maximum value is therefore 13.

Check this result as in the last example.

The values of the expressions could have been found directly from the formula $\dfrac{4ac - b^2}{4a}$.

Exercise XVII (a)

Draw the graphs of the following expressions, and find graphically the sum of the roots. Verify by calculation.

1. $x^2 + 5x + 6$. 2. $x^2 - 6x + 8$. 3. $-x^2 + 6x - 8$.

4. $-x^2 - 5x - 6$. 5. $x^2 + 6x + 2$. 6. $2x^2 + 2x + 1$.

Say whether the following expressions have maximum or minimum values, and find them in each case. Check your results by trial numbers.

7. $3x^2 + 12x - 2$. 8. $6 + 5x - 2x^2$. 9. $x - x^2$.

10. $4 - 3x^2$. 11. $3 + 2x^2$.

12. $2x^2 + 3x + 5$. (Notice that the roots of the equation $2x^2 + 3x + 5 = 0$ are imaginary.)

13. Applying the same reasoning to $3 - 4x + 0x^2$, what can you say about its values?

14. If the path of a projectile is represented by the equation $y = -\frac{1}{18}x^2 + \frac{2}{3}x$, what is its form? If x and y are in miles, determine

    (i) the greatest height to which the projectile rises, i.e. the maximum value of y;

    (ii) the horizontal range of the projectile, i.e. the distance between the values of x for which y is 0.

15. If the path of a projectile is given by the equation $y = 0{\cdot}4x - 0{\cdot}04x^2$, find (i) the horizontal range; (ii) the greatest height to which the projectile rises. (x and y are in miles.)

§ **4.** Another method of obtaining the roots of a quadratic equation from the graph of the expression is as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or

$$= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

We have seen that the axis of symmetry passes through

$$x = -\frac{b}{2a}.$$

And that for this value

$$ax^2 + bx + c = -\frac{b^2 - 4ac}{4a} \quad \text{or} \quad c - \frac{b^2}{4a}.$$

In the figures,

$$\text{ON} = -\frac{b}{2a} \quad \text{and} \quad \text{NM} = -\frac{b^2 - 4ac}{4a} \quad \text{or} \quad c - \frac{b^2}{4a},$$

also $$\text{OC} = \text{PN} = c.$$

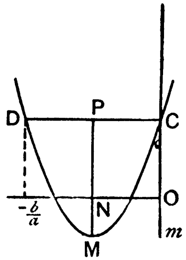


FIG. 7.

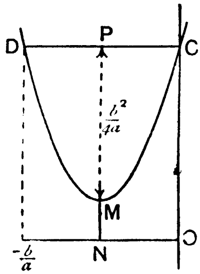


FIG. 8.

Now, PM = PN − NM in both figs. 7 and 8 (remember that in fig. 7, NM is negative).

$$\therefore \quad \text{PM} = c - \left(c - \frac{b^2}{4a}\right)$$
$$= \frac{b^2}{4a}.$$

You will notice that the roots are imaginary if PM < PN, i.e. if

$$\frac{b^2}{4a} < c \quad \text{or} \quad b^2 < 4ac,$$

which is the result mentioned on page 172.

From the measurements of ON, PM and NM, the roots may be written down as follows:

The first term is $-\frac{b}{2a}$, i.e. ON.

The second term is $\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\sqrt{\frac{-\text{NM}}{a}}.$

a can be found from PM and ON, thus:

$$\text{PM} = \frac{b^2}{4a} \quad \text{and} \quad \text{ON} = \frac{-b}{2a};$$

$$\therefore \ (\text{ON})^2 = \frac{b^2}{4a^2},$$

and

$$\frac{\text{PM}}{(\text{ON})^2} = \frac{\frac{b^2}{4a}}{\frac{b^2}{4a^2}} = \frac{b^2}{4a} \times \frac{4a^2}{b^2} = a.$$

$$\therefore \ \text{the second term is} \pm \sqrt{\frac{-\text{NM}}{\frac{\text{PM}}{(\text{ON})^2}}}.$$

Hence the roots are $x = \text{ON} \pm \sqrt{\dfrac{-\text{NM}}{\frac{\text{PM}}{(\text{ON})^2}}}.$

Or, simplifying $x = \text{ON}\left(1 \pm \sqrt{\dfrac{-\text{NM}}{\text{PM}}}\right).$

Observe that for real roots NM is negative, i.e. below the axis of x; and that for imaginary roots NM is positive, i.e. above the axis of x.

Example i.—The graph of a quadratic expression cuts the axis of y at 7 and has a minimum value -9, when x is 4. Find the roots of the equation, expression $=0$, and the expression.

$$x = \text{ON} \pm \sqrt{\frac{-\text{NM}}{\frac{\text{PM}}{(\text{ON})^2}}}$$

$$= 4 \pm \sqrt{\frac{-(-9)}{\frac{16}{(4)^2}}}$$

$$= 4 \pm 3 = 7 \text{ or } 1.$$

The expression is of the form $ax^2 + bx + c$.

In this case, $a = \dfrac{16}{(4)^2} = 1$, $\dfrac{-b}{2a} = 4$ from which $b = -8$ and $c = 7$.

Hence the expression is $x^2 - 8x + 7$.

Example ii.—A quadratic graph intersects the axis of Y at 50, and has the minimum value 18, the corresponding value of

x being 4. Find the roots of the equation, expression $= 0$, and the expression.

The roots are:

$$x = \mathrm{ON} \pm \sqrt{\frac{\dfrac{-\mathrm{NM}}{\mathrm{PM}}}{(\mathrm{ON})^2}}$$

$$= 4 \pm \sqrt{\frac{\dfrac{-18}{32}}{16}},$$

$$x = 4 \pm \sqrt{-9} \text{ or } 4 \pm 3\sqrt{-1}.$$

Notice that these roots are imaginary.

Since $a = \dfrac{32}{(4)^2} = 2$, $\dfrac{-b}{2a} = 4$, and $b = -16$, and $c = 50$, the expression is $2x^2 - 16x + 50$.

Observe that the method $(x - \text{one root})(x - \text{the other root})$ does not give the correct expression.

Exercise XVII (b)

1. Without solving the equations, state whether the following have real or imaginary roots:

   (i) $x^2 + x + 3 = 0$. (ii) $x^2 + x = 3$.

   (iii) $2x^2 + 3x + 2 = 0$. (iv) $5 - 2x - x^2 = 0$.

   (v) $4 - 2x^2 = 5x$. (vi) $3x^2 - 4x + 2 = 2x$.

2. The minimum value of a quadratic expression is 6, 3 being the corresponding value of x, and when x is 0 the value is 21. Find the expression and the roots of the equation, expression $= 0$.

3. When x is 0, the value of a quadratic expression is -48. When x is 3 the expression has its minimum value, namely -75. Find the expression.

4. Find the value of y in terms of x from the equation

   $$3x^2 + 2xy + y^2 = 1,$$

   and say for what values of x, y is imaginary.

   Graph a few of the real values of y.

5. If α and β are the roots of the equation $x^2 - ax + b^2 = 0$, find without solving the equation the value of $\alpha^3 - \beta^3$, of $\alpha + \beta$ and of $\alpha\beta$.

6. If α and β are the roots of the equation $ax^2+bx+c=0$, show that $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Find the value of $\frac{\alpha}{\beta}-\frac{\beta}{\alpha}$ in terms of a, b and c.

Find the equation in its simplest form whose roots are

$$2(-1+\sqrt{-3}) \text{ and } (-1+\sqrt{-3})^2.$$

7. If α and β are the roots of the equation $3x^2-5x+1=0$, find the equation whose roots are $\frac{\alpha}{1+\beta}$ and $\frac{\beta}{1+\alpha}$.

§ 5. Quadratic Form.

Any equation which has the form $ax^{2n}+bx^n+c=0$ can be solved by the methods given.

EXAMPLE.—$2x^4-3x^2-36=68$ (note that x^4 is the square of x^2),

$$x^4-\tfrac{3}{2}x^2=\tfrac{104}{2}.$$

Completing the square,

$$x^4-\tfrac{3}{2}x^2+(\tfrac{3}{4})^2=\tfrac{104}{2}+\tfrac{9}{16}=\tfrac{841}{16};$$

$$\therefore\ x^2-\tfrac{3}{4}=\pm\tfrac{29}{4}$$

and

$$\therefore\ x^2=\tfrac{3}{4}\pm\tfrac{29}{4}$$

$$=8 \text{ or } -\tfrac{26}{4}.$$

x is now found by taking the square root of 8. The other root is, of course, imaginary in this case.

$$\therefore\ x=\pm 2\sqrt{2}.$$

EXERCISE XVII (c)

Solve:

1. $x^4-20x^2+64=0$.
2. $x^4-13x^2+40=4$.
3. $ax^4+bx^2+c=0$.
4. $x^6-14x^3=-48$.
5. Draw the graph of x^4-13x^2+36, and determine how many times it cuts the axis of x, and also, how many turns or vertices it has.
6. By the method of testing factors (page 127), show that a is one root of the equation $x^3-3a^2x+2a^3=0$. Hence find the remaining roots.
7. Graph the expression x^3-x+2, and find how many times it cuts the axis of x, and how many vertices it has.

§6. Simultaneous Equations containing the Unknown to the Second Power.

TYPE I.—*Solved by substitution.*

(*a*) $$5x+2y=-4, \quad \text{(i)}$$
$$y=x^2+3x+5. \quad \text{(ii)}$$

Equation (i) contains the first power of x, only.

From (i), $$2y=-4-5x,$$
$$y=\frac{-4-5x}{2}. \quad \text{(iii)}$$

Substituting this value in equation (ii),

$$\frac{-4-5x}{2}=x^2+3x+5,$$
$$-4-5x=2x^2+6x+10,$$
$$2x^2+11x+14=0.$$

This is a quadratic equation, the roots of which are:

$$x=\frac{-11\pm\sqrt{121-112}}{4}$$
$$=-\tfrac{7}{2} \text{ or } -2.$$

From (iii), $$y=\frac{-4-5x}{2},$$
$$y=\tfrac{27}{4} \text{ or } 3.$$

(*b*) $$3x+2y=12, \quad \text{(i)}$$
$$2x^2-3y^2=-19. \quad \text{(ii)}$$

From (i), $$y=\frac{12-3x}{2}. \quad \text{(iii)}$$

Substituting in (ii),

$$2x^2-3\left(\frac{12-3x}{2}\right)^2=-19,$$
$$\tfrac{19}{4}x^2-54x+89=0,$$
$$x=2 \text{ or } -18\tfrac{14}{19}.$$

y is found from equation (iii).

(*c*) $$x+y=a, \quad \text{(i)}$$
$$xy=b. \quad \text{(ii)}$$

From (i), $$y=a-x. \quad \text{(iii)}$$

Substituting in (ii), $$x(a-x)=b. \quad \text{(iv)}$$

Equation (iv) is an ordinary quadratic.

The rest is easy.

TYPE II.—*All terms of the second degree. Solved by finding the ratio of the unknowns.*

(*a*) $$x^2 + y^2 = 164, \quad \text{.........................(i)}$$
$$xy = 80. \quad \text{.........................(ii)}$$

Let $y = kx$; then, from (i),
$$x^2 + k^2x^2 = 164; \quad \text{.........................(iii)}$$
from (ii), $$kx^2 = 80. \quad \text{.........................(iv)}$$

Dividing (iii) by (iv) and cancelling x^2,
$$\frac{1+k^2}{k} = \frac{164}{80}.$$

Cross multiplying, $$80 + 80k^2 = 164k,$$
$$80k^2 - 164k + 80 = 0,$$
or $$20k^2 - 41k + 20 = 0,$$
from which $$k = \frac{41 \pm \sqrt{(41)^2 - 1600}}{40}$$
$$= \tfrac{5}{4} \text{ or } \tfrac{4}{5}.$$

From (iv), x is found.

From (ii), y is found.

(*b*) $$2x^2 + 3xy + y^2 = 70,$$
$$6x^2 + xy - y^2 = 50.$$

Let $y = kx$, and proceed as in Example (*a*).

These equations can be solved graphically, by plotting the values of y against values of x, and finding the intersection points of the graphs. It will be found that the graph of the equation $x^2 + y^2 = a$, where a is a number, is a circle whose centre is the origin and whose radius is $\sqrt{a}$ (see equation, Type II (*a*)).

§7. Application to Surds.

From the identity $(\sqrt{x} \pm \sqrt{y})^2 = x + y \pm 2\sqrt{xy}$, the square root of an expression of the type $N \pm 2\sqrt{M}$ can be readily found; for it follows that

(i) $x + y = N$. (ii) $xy = M$.

From these simultaneous equations x and y can be determined.

EXAMPLE.—Find the square root of $20 - 8\sqrt{6}$.
$$20 - 8\sqrt{6} = 20 - 2\sqrt{96}.$$

Hence, if the root is of the form $\sqrt{x} - \sqrt{y}$,

(i) $x + y = 20$. (ii) $xy = 96$.

The values of x and y can be found from these simultaneous equations, but they are obviously 12 and 8. The square root is therefore $\pm(\sqrt{12} - \sqrt{8})$, which when simplified becomes,

$$\pm(2\sqrt{3} - 2\sqrt{2}) = \pm 2(\sqrt{3} - \sqrt{2}).$$

Exercise XVII (d)

1. Solve the example Type II (a) by forming two equations, one by adding and the other by subtracting twice equation (ii) from equation (i), and then extracting the square root of the resulting equations

Solve:

2. $2x - 5y = 3$ and $x^2 + xy = 20$.
3. $x - y = 3$, $x^2 + y^2 = 65$.
4. $\dfrac{x}{12} + \dfrac{y}{10} = x - y$, $\dfrac{7xy}{15} - \dfrac{2x}{3} = 2y$.
5. $9y - 2 = xy$, $xy + 2 = x$.
6. $x^2 + y^2 = 20$, $x + y = xy - 2$.
7. $x^2 + xy = 21 + 6y^2$, $xy - 2y^2 = 4$.

Find graphically the values of x for which the following expressions have the same values. Find these values.

8. $2x - 5$ and $x^2 - 3x + 1$.
9. $x^2 + 3x + 6$ and $-2x^2 - 9x - 3$.
10. $x^2 + 6x + 5$ and $12 + x - x^2$.
11. $\dfrac{1}{x}$ and $x^2 - x - 6$.

Solve:

12. $x^2 + y^2 = 5$, $3x + 4y = 2$.
13. $\dfrac{1}{x} + \dfrac{1}{y} = 1$, $6x + 2y = 1$.
14. $2x - y = 3$, $2x^2 + xy = 2$.
15. $3(x - 1)^2 + (y + 2)^2 = 9$, $x + y = 1$.
16. (i) $x^3 + y^3 = 189$, $x^2 - xy + y^2 = 21$. (ii) $3x^2 - 2xy - y^2 = 35$, $2x^2 + xy - 3y^2 = 0$.
17. $x + y = 5{\cdot}17$, $x^2 + y^2 = 14{\cdot}25$.

Find the square root of:

18. (i) $9 + 2\sqrt{14}$, (ii) $9 - 2\sqrt{14}$.
19. $23 - 6\sqrt{10}$.
20. $5 - 2\sqrt{6}$.
21. $36 + \sqrt{1292}$.

22. $2(a+\sqrt{a^2-b^2})$.

23. $\dfrac{a^2+b^2}{b(a-b)}-\sqrt{\dfrac{4a(a+b)}{b(a-b)}}$.

24. The difference between two numbers is 4, and the sum of their squares 106. Find them.

25. Divide 20 into two parts such that the sum of the squares of the parts is less by 8 than forty times one of the parts.

26. The perimeter of a rectangle is 34 inches, and its area 60 square inches; find the length of its diagonals.

27. Find a sector such that if its radius were increased by 5 inches its area would increase in the ratio of 3 to 2.

28. The distance in feet through which a body falls in a time t secs. is given by the equation

$$d=16t^2.$$

When a stone is dropped down the shaft of a mine the sound of its impact with the bottom is heard at the surface 10 seconds after the release of the stone.

If sound travels at the rate of 1120 feet per second, find the depth of the shaft.

29. Two men start at the same time for a town 75 miles distant, one cycling and the other by motor car. If the motorist travels 10 miles an hour faster than the cyclist and reaches the town 2 hours 40 minutes before him, find the rate at which each travels.

30. Draw and examine the graphs of $y=3\pm\sqrt{4-x^2}$.

Find (i) the range of x within which there are two values of y for a value of x.

(ii) The values of x for which there is only one value of y.

(iii) The ranges of the values of x for which there are no real values of y.

31. Draw and examine the graph of $\dfrac{x^2}{36}+\dfrac{y^2}{9}=1$.

What change would you make in this equation so that the graph would be the circumference of a circle?

REVISION EXERCISE II

1. The metal work of a Zeppelin is made of an alloy of density 3·2. If aluminium, density 2·6, and copper, density 8·9, are the constituent metals, find the ratio in which they occur.

2. Find the equation to the straight line passing through the points $(-4, 13)$ and $(5, -5)$.

3. What is the equation to the straight line which cuts the axis of y at -3, and makes an angle of $30°$ with the axis of x, the scales of the axes being alike?

4. Divide 75 into two parts such that one part is 9 more than twice the other.

5. If $E = C\sqrt{R^2 + p^2L^2}$, find C when $E = 161$, $R = 10$, $p = 40\pi$ and $L = 0{\cdot}1$.

6. (1) What factors of $x^4 - 5x^2 + 4$ are factors of $x^5 - 11x + 10$ also?
   (2) Factorise: (i) $x^4 - 6x^2y^2 - 16y^4$; (ii) $a^4 + 2a^3b - 2ab^3 - b^4$;
   (iii) $2x^3 - 3x^2y - 2x + 3y$; (iv) $2x^4 + 24x - 8x^2 - 48$.

7. Without multiplying the whole expressions, find the coefficient of x^2 in the product of $(x^2 - 3x + 2)$ and $(2x^2 + 5x - 3)$.

8. Simplify: (i) $3a\sqrt{2} - 2b\sqrt{3} + 2\sqrt{3} - 3\sqrt{2} + 5b\sqrt{3} + a\sqrt{2}$, and find its value when $a = 2$ and $b = -3$.
   (ii) $\dfrac{x^6 - y^6}{(x - y)(x^2 - xy + y^2)}$.

9. A number is exactly divisible by 3, if the sum of its digits is exactly divisible by 3. Prove this for a number of three digits, taking a, b and c to represent the digits. [Hint: $100a = (99a + a)$.]

10. Show that the cube of the sum of two numbers is equal to the sum of their cubes, together with three times their sum multiplied by their product.

11. Compute: $\pi\dfrac{(15{\cdot}83)^2 \times 0{\cdot}0385}{\sqrt[3]{10}} + 15{\cdot}83\pi\dfrac{26{\cdot}4\sqrt{5{\cdot}37}}{0{\cdot}852}$.

12. On the same diagram draw the graphs of $2x^2$ and $12 - 5x$, and find the values of x for which $2x^2 = 12 - 5x$.
    Apply what you have learnt to solve the equation $x^3 - 7x + 6 = 0$.

13. If $h = \frac{1}{2}gt^2$ and $v = gt$, show that $h = \dfrac{v^2}{2g}$.

14. Determine the equation to the parabola passing through the points, $(1, 8)$, $(-2, -7)$, and $(2, 5)$. Find also the co-ordinates of its vertex.

15. Show that the least value of $3x^2 - 6x + 5$ is 2.

CHAPTER XVIII

AREA BOUNDED BY A GRAPH, APPLICATIONS TO MENSURATION, AND SCIENCE

§ 1. Graphs.

The area bounded by a graph, the axis of X and two ordinates, has a definite significance.

To understand it, consider a graph parallel to the axis of X (fig. 1). The equation to the graph is $y = 0x + 5$.

Draw ordinates AB and CD at $x = 5$ and at $x = 15$; then the rectangle ABDC has a length, AC, of 10 units and an altitude of 5 units.

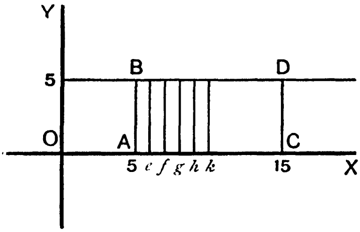


FIG. 1.

The area of ABDC is 50 units, the unit being the area of one of the small squares.

The area thus represents the product of 10 and 5.

The figure ABDC can be regarded as consisting of a number of narrow strips standing on the small bases Ae, ef, fg, gh, etc., and having the same altitude, namely AB.

Then, since the area ABDC is the sum of the areas of these strips,

$$\begin{aligned} ABDC &= AB \times Ae + AB \times ef + AB \,.\, fg + \text{etc.} \\ &= AB(Ae + ef + fg + \text{etc.}) \\ &= AB \,.\, AC. \end{aligned}$$

Now this is true no matter how small the parts on the axis of X may be.

The same is true in Arithmetic; for example, $3 \times 6 = 3(3 + 2 + 1)$, or 3(any set of numbers the sum of which is 6). **An area then may represent a product.**

§**2.** Take a straight-line graph inclined to the axis of X. Consider the area between two ordinates, AB and CD (fig. 2). In this case, not only does x change from OA to OC, but y also, from AB to CD.

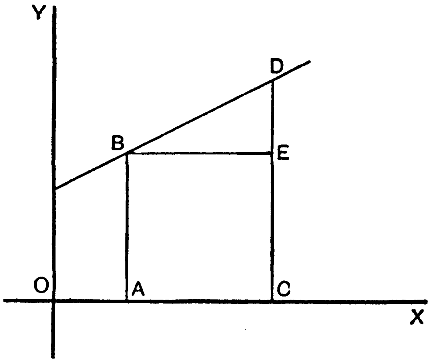


Fig. 2.

The area of the trapezoid ABCD may be regarded as made up of narrow strips, the strips being so narrow that they are practically rectangles. It will be noticed, however, that the altitude of the strips increases from AB to CD.

The area is the sum of these small areas, and as before represents a product, but in this case, of numbers which are changing.

Such a product, though difficult to determine by Arithmetic, is readily obtained by the graphical method.

The product of x and y, when x changes from OA to OC and y changes from AB to CD, is represented by the area of the trapezoid ABCD, which by Mensuration is:

$$\mathrm{AC} \times \frac{\mathrm{AB}+\mathrm{CD}}{2} = \mathrm{AC} \times \frac{\mathrm{AB}+\mathrm{CE}+\mathrm{ED}}{2}$$

$$= \mathrm{AC} \times \frac{2\mathrm{AB}+\mathrm{ED}}{2} = \mathrm{AC}\left(\mathrm{AB}+\frac{\mathrm{ED}}{2}\right).$$

The following examples show the importance of these results.

Applications.

(i) A body moves with a uniform speed of 16 feet per second. Construct a graph showing the relation between speed and time, and from it find the distance travelled in 20 seconds.

Represent time on the axis of X, and the speed in feet per second on the axis of Y (fig. 3). The graph is, of course, parallel to the axis of X.

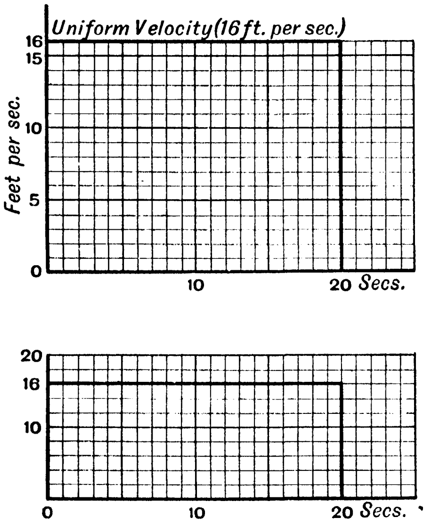


FIG. 3.

Now, distance is equal to the product of speed and time, and therefore if an ordinate is drawn from 20 on the axis of X, the area bounded by this ordinate, the graph, the axis of Y and the axis of X will represent the distance travelled in 20 seconds.

The distance is $16 \times 20 = 320$ feet.

From the two diagrams, it will be seen that this result does not depend upon the scales of the axes.

(ii) The work done by a force is measured by the product of the displacement and the force acting in the direction of the displacement. Show by a graph the work done when a force of 20 lbs. produces a displacement 12 feet in the direction of the force. This graph, like that in application (i), is a rectangle.

(iii) A body with an initial velocity of 10 feet per second gains velocity at the rate of 2 feet per second every second. Construct a graph showing the relation between velocity and time. From it find the relation between velocity and time, and also the distance travelled in 6 seconds.

The gradient of the graph is 2, and the added constant 10.

The equation is therefore of the form $y = 2x + 10$.

Or, if we call the velocity v and the time t, the equation becomes $v = 2t + 10$.

When t is 6 seconds, v is $(2 \times 6 + 10) = 22$ ft. per sec.

The ordinate at the point 6 on the axis of time represents 22 feet per second.

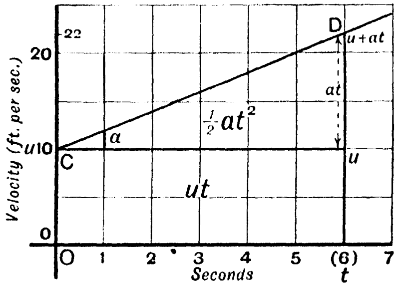


FIG. 4.

Draw this ordinate. Then, since distance is obtained by multiplying velocity by time, the area OCDt represents the distance travelled in the interval 0 to 6 seconds.

$$\text{Therefore the distance} = \frac{10 + 22}{2} \times 6 = 96 \text{ feet.}$$

Observe that:

(i) The gradient of the graph gives the rate at which the velocity changes. (Rate of change of velocity is called *acceleration.*)

(ii) The added constant is the initial velocity.

It is now easy to deduce the general formula for the distance travelled in a given interval of time by a body moving with uniform acceleration.

Let u = the initial velocity, a = the acceleration, i.e. the change in velocity in unit time, and t = the units in the interval of time, reckoned from the instant at which the velocity was u.

Then a is the gradient of the graph, u the added constant, and the velocity v at the end of the interval of time, $u + at$. (at is the total change in velocity.)

Drawing the ordinate at point t, its length represents $(u + at)$.

Hence the distance (d) travelled in the interval 0 to t is represented by the area OCDt.

Therefore, $$d = \frac{OC + tD}{2} \times Ot = \frac{u + (u + at)}{2} \times t$$
$$= ut + \tfrac{1}{2}at^2.$$

EXERCISE XVIII (A)

Special Cases and Examples.

1. If the body starts from rest, $u = 0$. Draw the graph for this case, and by it determine the formula for the distance covered in an interval 0 to t.

2. If the speed of the body is decreasing, the graph has a down gradient, i.e. the coefficient of t is negative.
   Draw a graph for this case, and deduce the formula for distance.

3. A falling body has a uniform acceleration of $+32$ ft. per sec. every second (approx.). Draw a graph showing the relation between velocity and time (the body starting from rest), and determine the distance covered in the following intervals of time:
   (i) During the first second.
   (ii) During the first 6 seconds.
   (iii) During the interval from the beginning of the fourth to the end of the ninth second.

4. Show on the general graph (fig. 4) that the average velocity during the interval 0 to t is the same as the actual velocity at the middle of the interval.

5. When a body is projected vertically from the earth it loses speed at the rate of 32 ft. per second every second. If the velocity of projection is 100 ft. per second, draw a graph showing how speed changes with time. From it find the formula for calculating the height to which the body will rise, and also the time taken.
   Complete the graph to represent the return journey also.

Work done by a Variable Force.

6. A wire rope weighing 10 lbs. per foot is coiled on the floor. It is gradually lifted vertically by taking hold of one end and raising it through a vertical height equal to the length of the rope. If the length is 60 feet, find the work done when the lower end just leaves the floor.

Plot the graph showing the force when the upper end is at various distances from the floor.

The work done is represented by the area of the triangle. Find also

(i) The work done in lifting the first half of the rope.

(ii) The work done in lifting the remaining half of the rope.

7. The following numbers were obtained in stretching a spring by hanging weights at its free end:

Increase in length in inches	0″	0·2″	0·4″	0·8″	1·0″	5·0″
Weight applied -	0	1 lb.	2 lbs.	4 lbs.	5 lbs.	25 lbs.

Draw a graph and find:

(i) An equation connecting the force and the elongation.

(ii) The work done in stretching the spring 1 foot at the same rate.

(iii) If the length of the unloaded spring is 2 feet, an equation connecting the length with the force applied.

8. The following numbers give the current in a short-circuited armature for different field currents:

Field current	0	0·75	0·163	0·253	0·342	x
Arm. current	14	45·4	85·5	125·6	165·6	y

Find the equation connecting armature current and field current.

§3. The relation between the graph of "velocity," and that of "positions" of a body moving with uniform acceleration is specially interesting.

The graphs are shown one directly under the other in fig. 5.

The initial velocity is taken as 5 ft. per sec. and the acceleration as 2 ft. per sec. per second.

In the upper graph, area AO1B represents the distance, 6 ft., covered in 1 second. In the lower graph, length 1A represents the distance covered in 1 second. Similarly areas AO2C, AO3D, AO4E represent the distances covered in 2, 3 and 4 seconds

respectively, and in the lower graph, these distances are represented respectively by the lengths 2B, 3C, 4D. Thus it is seen that the ordinates of the graph of positions represent the areas of the graph of velocity.

It will be readily understood that the difference between the ordinates 3C and 2B is equivalent to the area C23D.

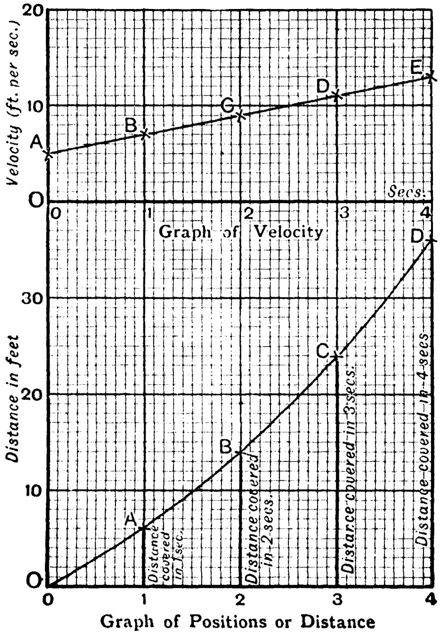


FIG. 5.

The lower graph is often called the **integral** of the upper graph.

§4. Area bounded by the Parabolic Graph.

In Mensuration you have learnt that the volume of a prism is obtained by multiplying the area of the base by the height. There is another way of expressing this rule.

Take a square prism and imagine a number of sections all parallel to the base to be made. These sections, called *right sections*, have the same shape and area as the base. Observe that they are perpendicular to the line of altitude.

The rule for volume may be expressed as the product of the altitude and the area of the section perpendicular to the altitude.

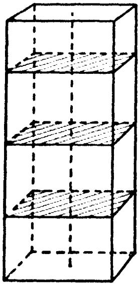


FIG. 6.

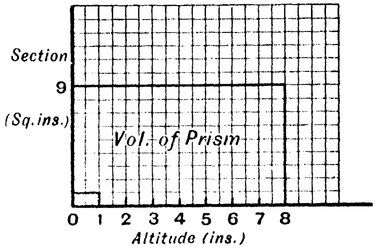


FIG. 7.

If we plot the area of section and the altitude at which the section is made for a square prism of altitude 8 inches and side of base 3 inches (fig. 7), the graph is a horizontal straight line. The area enclosed by the graph, ordinates at points 0 and 8 on the axis of X, and by the axis of x or altitude, represents the product of section and altitude, and therefore the **volume** of the prism.

In this case, the area representing the volume is a rectangle (8×9). The volume is therefore 72 cubic inches.

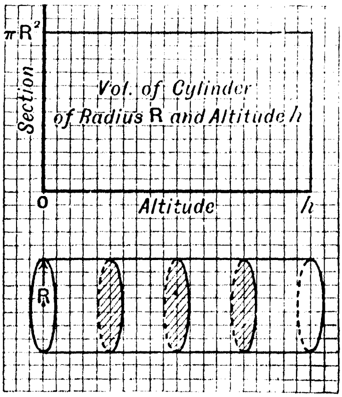


FIG. 8.

If we take a cylinder, the right sections are again equal to the area of the base. The graph of sections is therefore a horizontal straight line. Again, the area representing the volume is a rectangle.

§ **5.** Now consider a square pyramid. Like the square prism, its right sections are squares, but they increase in area from the apex to the base.

Now, it is known that the length of the edge of a section is proportional to its distance from the apex. Thus, taking a pyramid of base 12 ins. edge, and altitude 8 ins., the edge and area of section at distances from the apex are as follows:

Distance from Apex.	Edge of Section.	Area of Section.
0 inches.	0 inches.	0 sq. inches.
2 ,,	3 ,,	9 ,,
4 ,,	6 ,,	36 ,,
6 ,,	9 ,,	81 ,,
8 ,,	12 ,,	144 ,,

Plot the area of section against distance from the apex. The graph is a parabola. The area bounded by the graph represents the volume of the pyramid (fig. 9).

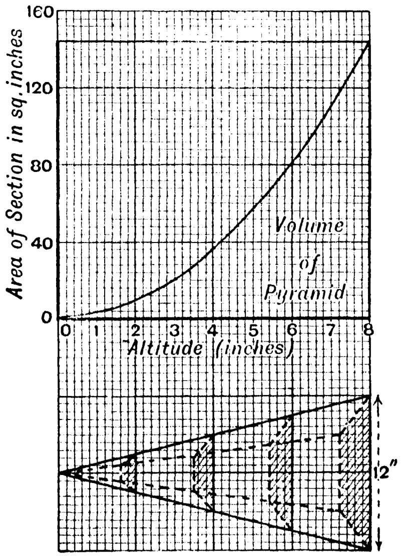


FIG. 9.

A more advanced student would be able to calculate this area from the equation to the graph. It can be obtained, however,

by counting the squares, or, better, by cutting out the figure in metal or cardboard, and comparing its weight with that of a sheet of known area. The result is 384 units, and represents 384 cub. ins.

On the same axes represent the volume of the prism having the same base and altitude, and compare the volumes. You will find that the volume of the pyramid is one-third that of the prism.

It is worth remembering that the area enclosed by the parabola, axis of X, *and the end ordinate is one-third of the rectangle having this portion of the axis of* X *as length and this ordinate as height. Observe that the vertex of the parabola is at the origin.*

§ **6.** In the same manner the volume of a cone can be compared with the volume of the cylinder having the same base and altitude. The right sections are circles of increasing radii, any radius being proportional to its distance from the apex.

Take a cone of, say, height 8 cms. and diameter 12 cms.

Tabulate the areas of sections at different distances from the apex, thus:

Distance from Apex.	Section.		
	Diameter.	Radius.	Area.
0 cms.	0 cms.	0 cms.	0 sq. cms.
2 "	3 "	1·5 "	$2{\cdot}25\pi$ "
4 "	6 "	3 "	9π "
6 "	9 "	4·5 "	$20{\cdot}25\pi$ "
8 "	12 "	6 "	36π "

In plotting the numbers there is no need to substitute the actual value of π. Scale the áxis of area (Y) in terms of π as shown in fig. 10.

You will find on plotting the area of section against the distance from the apex, that the graph is a parabola. On the same axes, draw the graph for the corresponding cylinder. The areas bounded by these graphs to the ordinate at $x=8$ represent the volumes of the cone and the cylinder respectively. On determining these, you will find that the volume of the cone is one-third that of the cylinder,

$$\text{i.e.}\quad \text{vol. of cone} = \frac{36\pi \times 8}{3} = 96\pi \text{ c.cms.}$$

Hence the rule :

The volume of a cone is one-third the product of the area of the base and the altitude.

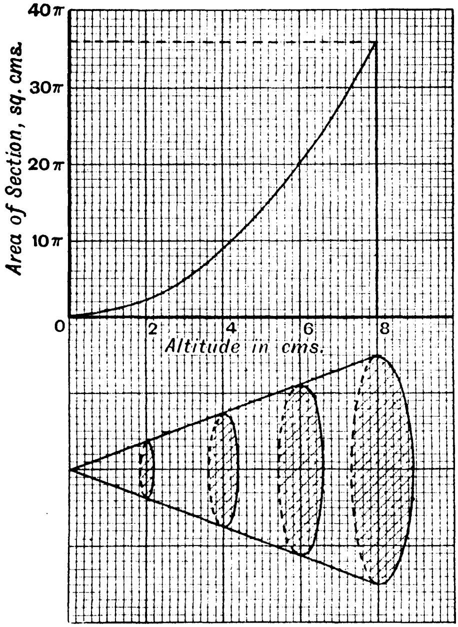


FIG. 10.

§ 7. The Sphere.

If sections are made at right angles to a diameter of a sphere, a number of parallel circles is obtained.

The areas of these circles vary.

Referring to fig. 11, consider the section at a point C a distance x from the end A of a diameter AB.

Let r be the radius of this section, and R the radius of the sphere.

Then $CD = r$ and $CB = (2R - x)$.

It has been established that $(CD)^2 = AC \cdot CB$,

$$\text{i.e.} \quad r^2 = x(2R - x).$$

Now the area of section is πr^2, which, by the above, is equal to:

$$\pi x(2R - x) = 2\pi Rx - \pi x^2.$$

Since this expression contains x^2, we conclude that the graph of section and distance is a parabola. Moreover, since the sign of x^2 is negative, and the expression contains x, the vertex is upwards and situated to the right of the axis of Y.

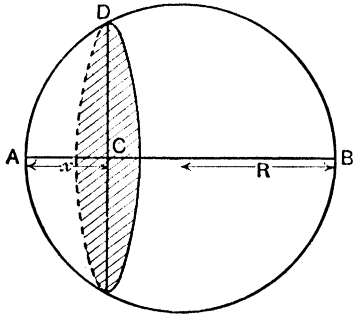


FIG. 11.

Plot the graph for various values of x, i.e. for points along AB, and the area bounded by it will represent the volume of the sphere.

Take a sphere of, say, 10 cms. diameter.

Tabulate as follows:

x, cms.	(Radius of Section)^2 =product of segments of diameter.	Area of Section, sq. cms.
0	$0 \times 10 = 0$	0
1	$1 \times 9 = 9$	9π
2	$2 \times 8 = 16$	16π
3	$3 \times 7 = 21$	21π
	etc.	

The graph is shown in fig. 12.

Through the vertex G draw the straight line EGF parallel to the axis of X.

Draw the ordinate 5G, which represents the section through the centre of the sphere.

It is readily seen that the areas EGA and GFB are each one-third of half the rectangle AEFB, or one-sixth of the whole rectangle.

It follows that the area enclosed by AGB and the axis of X (AB) is two-thirds of the rectangle AEFB.

Hence

$$\begin{aligned}
\text{Vol. of sphere} &= \tfrac{2}{3}\,\text{AB}\,.\,\text{AE}\\
&= \tfrac{2}{3} \times 2\text{R} \times (\text{area of centre section of sphere})\\
&= \tfrac{2}{3} \times 2\text{R} \times \pi\,\text{R}^2\\
&= \tfrac{4}{3}\,\pi\,\text{R}^3.\\
\text{In this case} &= \tfrac{4}{3}\,\pi\,(5)^3\\
&= \tfrac{500}{3}\,\pi \text{ c.cms.}
\end{aligned}$$

This graph is still more interesting.

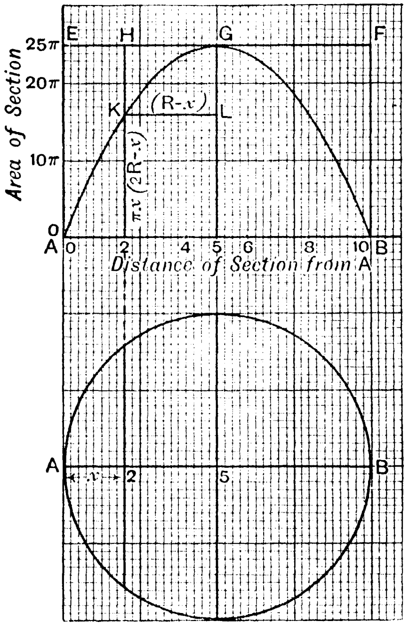


FIG. 12.

The area of the figure from 0 to the ordinate $x = 2$ represents the volume of the cap of the sphere cut off at the section passing through the point 2 cms. from A.

Similarly, the area between any two ordinates represents the volume of the zone between the two corresponding sections of the sphere.

Let us find the volume of the zone between the sections at **2** and **5**.

Draw the ordinates at $x=2$ and $x=5$, and produce the former to cut the straight line EGF at H.

From the point K, at which the ordinate **2**H cuts the graph, draw KL parallel to the axis of X to cut **5**G at L.

Then Area of figure KG**52** = fig. KGL + rect. KL**52**

$= \frac{2}{3}$ rect. HGLK + rect. KL**52**.

Since **2**K represents 16π and **5**G, 25π,

KH represents $(25\pi - 16\pi) = 9\pi$.

Hence Area of KG**52** $= \frac{2}{3} \times 9\pi \times 3 + 16\pi \times 3$

$= 18\pi + 48\pi$

$= 66\pi$ (units of area),

i.e., volume of the zone between the sections at 2 and 5 = 66π c.cms.

Further, the area AK**2** represents the volume of the cap of the sphere. This cap is the difference between the hemisphere and the zone just found,

i.e. the vol. of the cap $= (\frac{500}{6}\pi - 66\pi)$ c.cms.

$= 17\frac{1}{3}\pi$ c.cms.

You will find that the area of the figure AK**2** is $17\frac{1}{3}\pi$ units.

The general expressions for the volume of a zone, one plane surface of which is the section at the centre of the sphere, and for the volume of a cap can be deduced readily from the graph.

Let A**2** $= x$; then the area of the section at $x=2$ is $\pi x(2R - x)$.

Hence **2**K $= \pi x(2R - x)$ and KL $= (R - x)$.

Now **5**G = **2**H $= \pi R^2$ and KH = (**2**H − **2**K).

$\therefore$ KH $= \{\pi R^2 - \pi x(2R - x)\}$.

Again,

Area **2**KG**5** = rect. **2**KL**5** + fig. KGL

= rect. **2**KL**5** + $\frac{2}{3}$ rect. KHGL

= KL × **2**K + $\frac{2}{3}$KL × KH

= KL(**2**K + $\frac{2}{3}$KH)

$= (R - x)[\pi x(2R - x) + \frac{2}{3}\{\pi R^2 - \pi x(2R - x)\}]$

$= \pi(R - x)[x(2R - x) + \frac{2}{3}\{R^2 - x(2R - x)\}]$

$= \pi(R - x)[\frac{1}{3}x(2R - x) + \frac{2}{3}R^2]$

$= \frac{\pi(R - x)}{3}[2R^2 + 2Rx - x^2]$ ....(i)

If t is the altitude or thickness of the zone, $t=(R-x)$, from which $x=(R-t)$.

$$\text{Hence} \quad \text{Area } 2KG5 = \frac{\pi t}{3}[2R^2 + 2R(R-t) - (R-t)^2]$$

$$= \frac{\pi t}{3}(3R^2 - t^2),$$

$$\text{i.e. the volume of the zone of thickness } t = \frac{\pi t}{3}(3R^2 - t^2). \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Volume of the Cap.

$$\text{Fig. } AK2 = \text{fig. } AG5 - \text{fig. } 2KG5$$

$$= \tfrac{2}{3}\pi R^3 - \frac{\pi(R-x)}{3}[2R^2 + 2Rx - x^2], \text{ from (i)},$$

$$= \tfrac{1}{3}\pi[2R^3 - (R-x)\{2R^2 + 2Rx - x^2\}]$$

$$= \tfrac{1}{3}\pi[2R^3 - 2R^3 - 2R^2x + Rx^2 + 2R^2x + 2Rx^2 - x^3]$$

$$= \tfrac{1}{3}\pi x^2(3R - x).$$

Hence

$$\text{Volume of the cap of altitude } x = \tfrac{1}{3}\pi x^2(3R - x). \quad \ldots\ldots\text{(iii)}$$

Observe that results (ii) and (iii) both reduce to the volume of a hemisphere when $t=R$ and $x=R$.

By a similar process, the general expression for the volume of the part remaining when a cone is cut from a larger cone by making a right section can be established.

The part remaining is called a frustum.

The following hints will help you.

If the altitude of the whole cone is h and of the frustum t, then the altitude of the cone cut off is $(h-t)$.

The area on the graph diagram (fig. 13) representing the volume of the frustum is EDCB, and of the cone cut off, ADE.

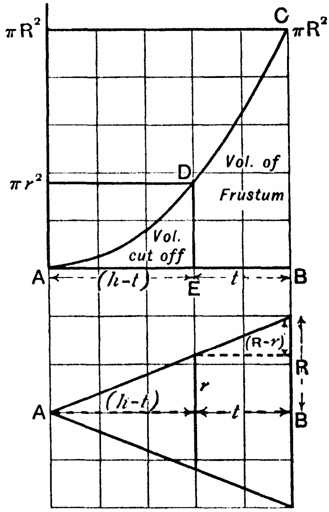


FIG. 13.

$$\text{Area } EDCB = \text{area } ACB - \text{area } ADE$$

$$= \tfrac{1}{3}AB \times BC - \tfrac{1}{3}AE \times DE.$$

If r and R are respectively the radii of the top and bottom of the frustum, then, by similar triangles,

$$\frac{h}{R}=\frac{t}{R-r} \quad \text{and} \quad \frac{h-t}{t}=\frac{r}{R-r}.$$

Substitute for h and $(h-t)$, and the final result should be,

$$\text{Vol. of frustum} = \frac{\pi t}{3}(R^2+Rr+r^2).$$

Area of the Curved Surface of a Spherical Cap.

Compare the formula for the area of a triangle, viz. half the product of the base and the altitude, with that for the area of a sector of a circle, viz. half the product of the arc and the radius. If the arc of the sector is called the base, and the radius the altitude, the two formulae become the same.

A similar comparison can be made between the rules for determining the volume of a cone having a plane base and a cone having a spherical base, the centre of curvature of which is the apex of the cone.

In both cases, the volume is one-third the product of the area of the base and the altitude. From this rule, the area of the curved surface of a spherical cap can be determined.

Referring to the figure,

Vol. of "spherical" cone = Vol. of cap + vol. of "plane" cone

$$= \tfrac{1}{3}\pi x^2(3R-x) + \tfrac{1}{3}\pi c^2(R-x)$$

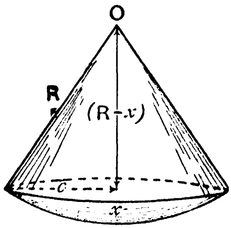


FIG. 14.

(substituting $x(2R-x)$ for c^2)

$$\begin{aligned} &= \tfrac{1}{3}\pi x^2(3R-x) + \tfrac{1}{3}\pi x(2R-x)(R-x) \\ &= \tfrac{1}{3}\pi x\{x(3R-x)+(2R-x)(R-x)\} \\ &= \tfrac{1}{3}\pi x(3Rx-x^2+2R^2-3Rx+x^2) \\ &= \tfrac{2}{3}\pi R^2 x. \end{aligned}$$

Now $\dfrac{\text{Area of base} \times \text{altitude}}{3} = \text{vol. of "spherical" cone};$

$$\therefore \text{ area of base} = \frac{3 \times \text{vol. of "spherical" cone}}{\text{altitude}}$$

$$= \frac{3 \times \frac{2}{3}\pi R^2 x}{R}$$

$$= 2\pi R x.$$

A rigorous proof of this formula is given in Chapter XXIV.

In the case of the hemisphere, $x = R$, and the formula becomes $2\pi R^2$. For the surface of the whole sphere, the formula is, of course, $4\pi R^2$; that is, four times the area of the circle of the same radius.

EXAMPLE.—To find the area of the earth's surface within the horizon of an observer at a height h.

Referring to the figure, by Geometry,

$$H^2 = h(h + D), \text{ where D is the earth's diameter.}$$

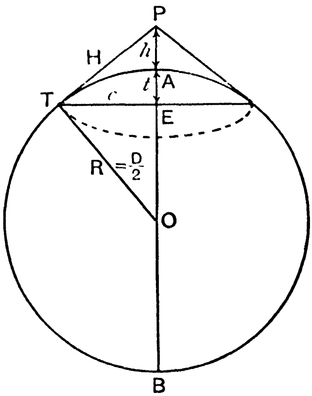


FIG. 15.

From the right-angled triangle PET,

$$c^2 + (h + t)^2 = h(h + D),$$

$$t(D - t) + (h + t)^2 = h(h + D),$$

$$Dt - t^2 + h^2 + 2ht + t^2 = h^2 + Dh,$$

from which $$t = \frac{Dh}{D + 2h}.$$

If, as is usual, h is small compared with D, this equation reduces to

$$t = h.$$

The area of the surface of the spherical cap bounded by the circle through T is $\pi \mathrm{D}t$.

Hence

$$\text{Area within the observer's horizon} = \pi \mathrm{D}t$$
$$= \pi \mathrm{D}h \text{ (approx.)}.$$

Note.—When h is small compared with D, the arc AT is approximately equal to the tangent PT. Hence, by Chapter XVI, page 176, the range of vision (AT) is $\sqrt{\mathrm{D}h}$.

EXAMPLE.—Find the area of the earth's surface within the horizon of an observer in a Zeppelin $1\frac{1}{2}$ miles above the earth.

$$\text{Area} = \pi \mathrm{D}h = 3{\cdot}14 \times 8000 \times 1\tfrac{1}{2} \text{ sq. miles}$$
$$= 37{,}700 \text{ sq. miles (approx.)}.$$

The range of vision in this case is

$$\sqrt{8000 \times 1\tfrac{1}{2}} = \sqrt{12000}$$
$$= 110 \text{ miles (approx.)}.$$

EXERCISE XVIII (B)

1. Find the ratio of the volume of a sphere to that of the circumscribing cylinder, i.e. of a cylinder of diameter and height equal to the diameter of the sphere.

2. From the graph relating to the cone (page 201), find the area of section, and then calculate the radius of a cylinder of the same altitude and volume as the cone.

3. From the graph relating to the sphere, find the area of section, then calculate the radius of a cylinder of equal volume and of altitude equal to the diameter of the sphere.

4. Find by graph the volume of the part of the cone left when a cone of half the altitude of the full cone is cut off.

5. On the figures showing the graphs of sections of the sphere and a cone, construct rectangles, the areas of which are equal to the areas representing the volumes of these solids.

   The altitude of each rectangle represents the mean section of the solid.

   Find, in each case, the position at which the actual section of the solid is equal to the mean section.

6. Take a spherical flask, measure its diameter, length and width of neck, etc., and calculate its volume. Verify your result by filling it with water and measuring the quantity.

7. Procure a conical flask, determine its dimensions and calculate its volume. Verify as in the last case.

8. A cylindrical log of wood, 10 inches in diameter and 6 feet long, is planed down on one side until a plane surface of width 3 inches is obtained. Find (i) the girth of the reduced log, (ii) the volume remaining.

9. Find the curved surface of the cap of altitude 3 inches, cut from a hemisphere of radius 8 inches. What is the area of the curved surface of the remaining zone?

10. Establish a formula for the curved surface of the zone of a sphere.

11. Calculate the area of the Arctic cap of the earth, and of one of the Temperate zones.

12. In testing the efficiency of an electric motor, the following numbers were found for the current and the efficiency:

Current -	0	2	5	10	12·5	15	20 amps.
Efficiency (%)	0	24·8	53·6	80·4	83·75	80·4	53·6

Plot the numbers, and find an equation connecting efficiency and current.

13. The power lost in a motor depends upon the speed. The following numbers were obtained in an experiment:

Speed (revs. per min.)	0	100	200	300	400	500	600	800	1000
Watts lost - - -	0	20	48	75	110	155	200	310	430

Find the law.

14. The following numbers give the velocity of a falling body after it has fallen through the given distances. Find the law connecting the two quantities.

Distance (ft.), d - -	0	2	4	9	16	25	36	64	100
Velocity (ft. per sec.), v	0	11·3	16	24	32	40	48	64	80

15. The volumes of square pyramids of the same altitude but of different bases are as follows:

Edge	2	3	4	5	6	8
Volume	16	36	64	100	144	256

Find the relation between volume and edge.

16. The following numbers were obtained when finding the volume of cones of different radii of base but of the same height, viz. 5 cms. by displacement of water:

R (cms.)	0	1	2	3
V (c.c.)	0	5·25	20·95	47·25

17. The following rises in temperature are obtained after the currents shown are passed for the same time through a coil of wire placed in a quantity of water. Find the law.

Current (C) - -	0	2	3	4	6	8
Rise in temperature (θ)	0	6	$13\frac{1}{2}$	24	54	96

18. The following numbers give the distance through which a body falls from rest in various intervals of time:

Time (secs)., t	0	1	2	3	4
Distance (ft.), d	0	16	64	144	256

Plot these numbers, and from the graph deduce the law connecting d and t.

19. In determining the meridian and latitude, the following numbers were obtained. They seem almost to follow the parabolic law. See if they do.

Azimuth	0°	5°	10°	15°	20°	25°	30°	35°	40°
Altitude	36·85°	37·6°	38·1°	38·45°	38·55°	38·45°	38·15°	37·65°	36·9°

20. The following numbers show the weight per mile of wire of the same material but of different diameters.

Find the equation connecting them.

Diam. (ins.)	·010	·040	·080	·160	·192
Weight (lbs.)	1·598	25·58	102·3	409·2	598·1

21. The table shows the available power from an electric generator when supplying the various currents given:

Power (watts)	72	128	192	168	128
Current (amps.)	2	4	8	14	16

Find the law connecting power and current, and the value of the current for which the power is a maximum.

22. A cone 10 cms. high floats, apex downwards, in water, and its apex is at such a depth that half its volume is beneath the surface. Find how far the apex is below the surface.

23. A hollow tin cone (diameter of base, 12 ins.; altitude, 12 ins.), when placed, apex downwards, in water floats with its apex 8 ins. below the surface.

How much farther would it sink if water were poured into the tin cone to a height of 6 ins.?

(Remember that a floating body displaces a volume of liquid whose weight is equal to that of the body.)

24. A hollow metal sphere, diameter 5 cms., having a small hole in it, floats in water so that its lowest point is 4 cms. below the surface. Find its weight. How much water must be poured into the sphere to make it just sink below the surface? Find, graphically or otherwise, the greatest depth of the water in the sphere.

(The thickness of the material of the sphere may be neglected.)

25. Sixty-four balls are arranged in the form of a cube inside a cubical box into which they just fit, the layers being similar. Water is now poured into the box, and it is found that just 800 cubic inches are required to fill it. Find the diameter of a ball.

26. A donkey is tethered to a cylindrical post 1 foot in diameter, by a rope 16 feet long.

Keeping the rope taut, and starting from the position such that lapping begins at once, the donkey, by walking round and round the post, laps the rope upon it.

(1) Find the number of revolutions made.

(2) Plot a graph showing the length of the rope not wound upon the post for various angles the rope turns through. What does the area of this graph indicate?

(3) Find the distance travelled by the donkey in this spiralic path.

(4) Determine the area swept over by the rope during the first revolution.

Construct the curve geometrically to scale, and, using the transparent paper or other method, check (3).

27. Two donkeys, tethered to the same post by equal ropes, walk round the post in opposite directions, the ropes being kept taut. If they start and meet as indicated in fig. 16,

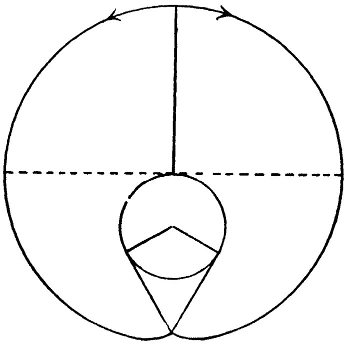


FIG. 16.

find, graphically or otherwise, how far each walks before they meet, given that the point of meeting is 3 feet from the post, and the radius of the post is 1 foot.

Find also the area enclosed by their paths.

28. The gas-bag of an airship has the shape of a cylinder with hemispherical ends. If the total length is 220 feet and the width 20 feet, find the volume.

If the bag is filled with hydrogen gas, one cubic foot of which weighs 0·00558 lb., find the total weight of the contents.

Calculate the weight of air the bag displaces, if air is 14·4 times as heavy as hydrogen.

Neglecting the weight and thickness of the bag, the difference between these weights is the "lift" of the bag. What is it in this case?

CHAPTER XIX

TRIGONOMETRY, APPLICATIONS TO MECHANICS, ETC.

§ 1. Trigonometry.

Trigonometrical Ratios and their Relation.

It is important to note how powers of the trigonometrical ratios are written. E.g. the square of sin A is written $\sin^2 A$.

Again,
$$\sin 45^\circ = \frac{1}{\sqrt{2}};$$
$$\therefore \quad \sin^2 45 = \frac{1}{2}.$$

Referring to the right-angled triangle ABC (fig. 1),
$$a^2 + b^2 = c^2.$$

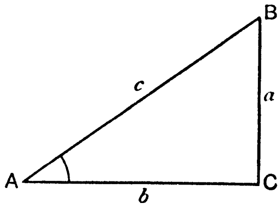


FIG. 1.

Divide all through by c^2; then
$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$
or
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1;$$
$$\therefore \quad \sin^2 A + \cos^2 A = 1.$$

EXERCISE.—(1) Arrange this for finding:

(i) sin A in terms of cos A.

(ii) cos A in terms of sin A.

(2) Choose any angle, and from the values given in the tables verify this relation.

§ 2. Names of the Inverse of the three Trigonometrical Functions given.

The inverse of the sine ratio is called the *cosecant*—written shortly *cosec*. Thus, referring to fig. 1, cosec A is $\frac{c}{a}$.

The inverse of the cosine ratio is called the *secant*—written shortly *sec*. Thus sec A is $\frac{c}{b}$.

The inverse of the tangent ratio is called the *cotangent*—written shortly *cot*. Thus cot A is $\frac{b}{a}$.

Summarise these statements in the form cosec A $= \frac{1}{\sin A}$, etc.

§ **3.** Starting with the relation

$$a^2 + b^2 = c^2,$$

and dividing in the first instance by a^2, and in the second by b^2, find relations corresponding to that in § **1.** Your results should be:

(i) $\operatorname{cosec}^2 A - \cot^2 A = 1$. (ii) $\sec^2 A - \tan^2 A = 1$.

All these relations, and the method of establishing them, should be remembered.

EXERCISE XIX (A)

1. Making use of the inverse functions, find cosec 30°, sec 30°, cot 30°, and these functions of 45°, 60° and 90°.

2. Give the reason for the following equalities:

$$\sin^2 A + \cos^2 A = \operatorname{cosec}^2 A - \cot^2 A = \sec^2 A - \tan^2 A.$$

3. If $t = \frac{V^2}{g}\cos^2 A$, find V when t is 5, A 30° and g 32.

4. Express $\cos^2 A$ in terms of $\sec^2 A$, then convert $\sec^2 A$ into $\tan^2 A$, and so obtain $\cos^2 A$ in terms of $\tan^2 A$; continuing, obtain $\cos^2 A$ in terms of $\operatorname{cosec}^2 A$.

   Commencing again with $\cos^2 A$, obtain $\cos^2 A$ in terms of $\sin^2 A$, and from this in terms of $\operatorname{cosec}^2 A$.

   See that the two results agree.

5. Find $\sin A$ in terms of $\sec A$.

6. Find $\cos A$ when $\tan A = \frac{2}{3}$, and $\cos A$ when $\cot A = \frac{5}{4}$.

7. Show that

$$\text{(i)} \quad \sin^2 A + \tan^2 A = \sec^2 A - \cos^2 A.$$

$$\text{(ii)} \quad \sin^2 A(1 + \cos^2 A) = 1 - \cos^4 A.$$

8. In fig. 1, the angle B is equal to $(90° - A)$; hence find the trigonometrical functions of $(90° - A)$ in terms of the functions of A. E.g. $\operatorname{cosec} B = \operatorname{cosec}(90° - A) = \dfrac{c}{b} = \sec A$.

9. Find $\tan A$ in terms of (i) $\sin A$, (ii) $\cos A$.

10. Plot the values of the six trigonometrical ratios for various values of x, where x represents the angle from 0° to 90°.

§ **4.** The following applications to Mechanics are important:

(1) If OX and OY represent forces acting at a point O (fig. 2), then OR, the diagonal of the parallelogram OXRY, represents their resultant. (Lines like OX, OY and OR are called Vectors.)

From the forces OX and OY, and the angle between their directions, the value of OR can be calculated.

Draw RP perpendicular to OX, or to OX produced.

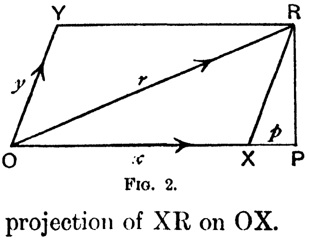


FIG. 2.

XP is called the projection of XR on OX.

Observe that $\qquad \angle RXP = \angle YOX.$

Now
$$\frac{XP}{XR} = \cos \angle RXP$$
$$= \cos \angle YOX;$$
$$\therefore \; XP = XR \times \cos \angle YOX. \qquad \text{(i)}$$

Let the forces be x and y, the angle between their directions A, the projection XP, p, and the resultant of x and y, r.

Then, by Geometry,

$$r^2 = x^2 + y^2 + 2xp$$
$$= x^2 + y^2 + 2xy \cos A^* \text{ (from (i), and since } XR = OY),$$
$$\text{i.e. } r = \sqrt{x^2 + y^2 + 2xy \cos A} \quad \text{.......... (ii)}$$

The angle ROX, which OR makes with OX, is determined as follows:

$$OP = OX + XP = x + y \cos A$$

and

$$\cos \angle ROX = \frac{OP}{OR} = \frac{x + y \cos A}{r},$$

i.e. the angle ROX is such that its cosine is equal to $\dfrac{x + y \cos A}{r}$.

The method of writing such a statement is $\cos^{-1}\left(\dfrac{x + y \cos A}{r}\right)$.

Thus, $\cos^{-1} \frac{1}{2}$ means, the angle whose cosine is $\frac{1}{2}$. What is this angle?

Example.—Find the resultant of forces of 10 lbs. and 6 lbs. acting at an angle of 50°, at the same point.

$$r = \sqrt{(6)^2 + (10)^2 + 2 \times 6 \times 10 \times \cos 50^\circ}$$
$$= \sqrt{136 + 120 \times \cdot 6428}$$
$$= \sqrt{136 + 77 \cdot 14}$$
$$= \sqrt{213 \cdot 14}$$
$$= 14 \cdot 6 \text{ lbs.}$$

The direction of r is such that the cosine of the angle which it makes with the direction of the force 6 lbs. is equal to

$$\frac{6 + 10 \cos 50}{14 \cdot 6} = \frac{6 + 10 \times \cdot 6428}{14 \cdot 6}$$
$$= \frac{12 \cdot 428}{14 \cdot 6}$$
$$= \cdot 8511.$$

From tables, $\cos^{-1} \cdot 8511 = 32^\circ$ (approx.),

i.e. the resultant makes an angle of 32° with the direction of the force 6 lbs.

* When A is obtuse, cos A is negative.

(2) The vector principle is applicable to velocity. An aeroplane travelling N.E. at 80 miles per hour in a calm area, suddenly enters an area in which a W. wind is blowing at 30 miles per hour. Find the resultant direction and speed.

(Remember, a west wind blows towards the east.)

From the figure:

$$\begin{aligned} r^2 &= x^2 + y^2 + 2xy \cos 45^\circ \\ &= 6400 + 900 + 4800 \times \cdot 707 \\ &= 7300 + 3393 \cdot 6 \\ &= 10693 \cdot 6 ; \\ \therefore \quad r &= 103 \cdot 4 \text{ miles an hour.} \end{aligned}$$

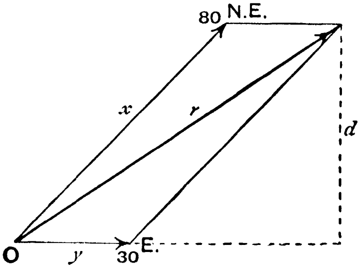


Fig. 3.

You are left to find the angle A either by the method on page 216 or by finding d and applying $\sin A = \frac{d}{r}$.

Note.—When x and y are determined from r, r is said to be ***resolved*** into its *components*. If the directions of x and y are at right angles to one another, and A is the angle between r and x, then

$$x = r \cos A, \quad \text{and} \quad y = r \sin A.$$

Draw a figure and verify these. Show also that $x^2 + y^2 = r^2$.

§ 5. Relative Velocity.

It was mentioned as early as in Chapter II. that in finding differences we may regard the zero as being moved to the number from which we have to view the difference. Relative velocities are determined in the same way.

Example i.—Two trains, A and B, are running in the same direction on parallel lines. If A goes at 30 miles an hour and B at 20 miles an hour, find the relative velocity of A to B, and of B

to A. In other words, find at what velocity A appears to be going to a person in B, and at what velocity B appears to be going to a person in A.

(1) If the velocity of B is regarded as the zero, then, considered from this zero, the velocity of A is $(30 - 20)$, i.e. 10 miles per hour, and is in the same direction as that of A.

(2) If the velocity of A is regarded as the zero, then relative to this zero the velocity of B is $(20 - 30)$, i.e. -10 miles per hour. The negative sign shows that, relative to A, the velocity of B is reversed. To a person in train A, train B appears to be going backwards at a rate of 10 miles per hour.

These statements are borne out by experience.

EXAMPLE ii.—Two trains, A and B, leave a junction, A travelling east at 20 miles per hour and B north-east at 30 miles per hour. Find their relative velocities.

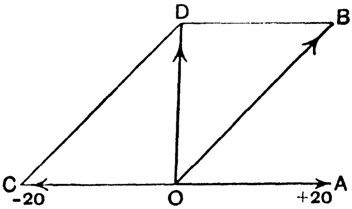


FIG. 4.

The relative velocity of B, regarded from A, is found as follows: OA and OB are drawn to represent in magnitude and direction the actual velocities of the trains.

The velocity of train A is reduced to the zero by impressing upon it a velocity equal, but opposite to that represented by OA. The same velocity is impressed upon OB, and the resultant found.

In the figure, OD is the resultant, OC being the impressed velocity. The resultant OD represents the velocity of B relative to A.

This is seen to be correct, for when train A reaches a point corresponding to A in the figure, train B will have a position corresponding to point B, and therefore a person in A must look in the direction AB, which is parallel to OD, to see train B.

The velocity of A with respect to B is found by drawing OB in the reverse direction, and impressing this reversed velocity on A and finding the resultant. Determine this.

§ 6. Projection of an Area.

Consider a rectangle ABCD, the plane of which makes an angle with the plane of the page of this book.

(This angle is measured between straight lines drawn one in each plane, from a point on the line of intersection of the planes, and at right angles to it.)

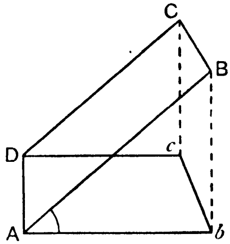


Fig. 5.

In the figure, AD is the line of intersection, and if Ab and AB are at right angles to AD, $\angle$BAb is the angle between the planes.

If Bb and Cc are drawn at right angles to the plane of the page, then AbcD is called the projection of ABCD on the plane of the page.

When ABCD is a rectangle, AbcD also is a rectangle.

Let angle BAb be denoted by A; then Ab = AB cos A.

Now Area of AbcD = AD × Ab

= AD × AB cos A

= area of ABCD × cos A,

i.e. the area of the projection of the area ABCD on the plane of the page is equal to the area of ABCD multiplied by the cosine of the angle between the planes.

It follows also that area of ABCD $= \dfrac{\text{A}bc\text{D}}{\cos \text{A}}$.

This relation is quite general. It is true for plane areas of all shapes.

§ 7. The Flight of a Projectile.

It has been mentioned already that the path or trajectory of a projectile is a parabola.

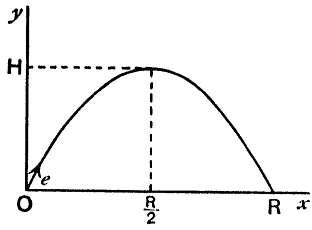


Fig. 6.

If the point of projection is the origin, then the equation of the path is of the form

$$y = ax^2 + bx \quad (\text{since } c \text{ is } 0)\ldots.(\text{i})$$

See fig. 6.

The coefficient of x^2 is of course negative.

If H is the greatest altitude attained, and R the horizontal range of the projectile, we know

from Chapter XVII, pages 179, 180, that

$$R = -\frac{b}{a} \quad \text{and} \quad H = -\frac{b^2}{4a}.$$

From these relations it readily follows that

$$b = \frac{4H}{R} \quad \text{and} \quad a = -\frac{4H}{R^2}.$$

Hence, in terms of the horizontal range and the maximum altitude, the equation to the path is

$$y = -\frac{4H}{R^2}x^2 + \frac{4H}{R}x. \quad \text{.....................(ii)}$$

Now H and R depend upon the angle of elevation of the gun and on the velocity of projection.

Let V = the velocity of projection in, say, feet per second, and e = the angle of elevation of the gun (more correctly, the 'quadrant' angle). Then, resolving the velocity, vertically and horizontally (see Note, § **4**),

Vertical velocity $= V \sin e$ and horizontal velocity $= V \cos e$.

If the vertical velocity is subject to an acceleration of $-g$ during ascent and g during descent (g is approximately 32 ft. per sec. per sec.), then the time to reach the greatest altitude is $\frac{V \sin e}{g}$ and the time of flight double this, namely $\frac{2V \sin e}{g}$.

Now the horizontal velocity is constant, and therefore

The horizontal range, R = horizontal velocity × time

$$= V \cos e \times \frac{2V \sin e}{g}$$

$$= \frac{2V^2}{g} \sin e \cos e.$$

Hence $$R = \frac{2V^2}{g} \sin e \cos e. \quad \text{.....................(iii)}$$

Again, from Ex. XVIII (A), 1,

$$H = \tfrac{1}{2}gt^2;$$

and since $$t = \frac{V \sin e}{g},$$

therefore $$H = \frac{V^2 \sin^2 e}{2g}. \quad \text{.............................(iv)}$$

Equation (ii) now becomes

$$y = \frac{\dfrac{-4V^2\sin^2 e}{2g}}{\dfrac{4V^4\sin^2 e\cos^2 e}{g^2}}x^2 + \frac{\dfrac{4V^2\sin^2 e}{2g}}{\dfrac{2V^2\sin e\cos e}{g}}x$$

$$= -\frac{g}{2V^2\cos^2 e}x^2 + \frac{\sin e}{\cos e}x,$$

or, since $\dfrac{1}{\cos^2 e} = \sec^2 e$ and $\dfrac{\sin e}{\cos e} = \tan e$,

$$y = -\frac{g}{2V^2}x^2\sec^2 e + x\tan e.^* \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

By means of this equation, the altitude (y) at any point of the flight can be determined.

§ 8. Duration of Day.

The following is a simple method of establishing the formula for the determination of the duration of day at any latitude:

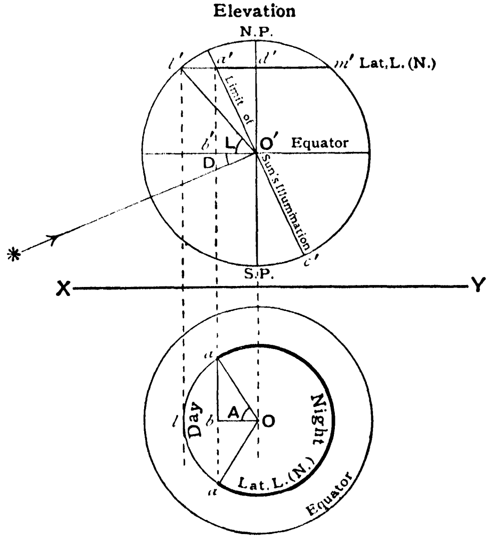


FIG. 7.

* A simpler proof of this equation is given in Chapter XXIV.

The figure is an elevation and plan of the earth, the season chosen being winter in North Latitude, the declination of the sun being therefore south.

Referring to the elevation, L is angle of the latitude (N.), D the angle of declination of the sun on the date considered (S.), $a'o'c'$ the elevation of the terminator or limit of the sun's illumination of the earth (at right angles to the line showing the direction of the sun's rays), a' is the point at which the terminator $a'o'c'$ cuts the elevation of the line of latitude L° N., say 52° N.

Referring to the plan, a and a are the points in plan corresponding to a' in elevation; O is the plan of the polar axis.

It is evident that a place on the circle of latitude L° N. will receive light while moving along the path represented by the arc ala (plan), and be in darkness while moving along the path represented by the arc ama.

Since these arcs are proportional to the angles, they subtend at the centre, then, if $\angle aoa$ (plan) is 2A°, we have:

$$\frac{\text{Duration of day}}{24 \text{ hours}} = \frac{2\text{A}^\circ}{360} = \frac{\text{A}^\circ}{180}.$$

Referring to the figure, it is seen that

$$a'd' \text{ (elev.)} = bo \text{ (plan)}, \qquad \angle o'l'd' \text{ (elev.)} = \text{L},$$

$$l'd' \text{ (elev.)} = oa \text{ (plan)}, \qquad \angle a'o'd' \text{ (elev.)} = \text{D}.$$

Now $o'd' = \text{R} \sin \text{L}$, and $l'd' = \text{R} \cos \text{L}$ (where R is the radius of the earth).

Again, $$a'd' = o'd' \tan \text{D}$$
$$= \text{R} \sin \text{L} \tan \text{D}.$$

Hence $$bo = \text{R} \sin \text{L} \tan \text{D}.$$

Now $\angle aob = \text{A}$ and $$\cos \text{A} = \frac{bo}{ao} = \frac{\text{R} \sin \text{L} \tan \text{D}}{\text{R} \cos \text{L}}$$
$$= \tan \text{L} \tan \text{D}.$$

Hence $\text{A} = \cos^{-1}(\tan \text{L} \tan \text{D})$, which determines A.

From (i), Duration of day $= \dfrac{\cos^{-1}(\tan \text{L} \tan \text{D})}{180} \times 24$ hours

EXAMPLE.—If $L = 52°$ N. and $D = 23\frac{1}{2}°$ (Dec. 22),

$$\text{Duration of day} = \frac{\cos^{-1}(\tan 52 \tan 23\frac{1}{2})}{180°} \times 24$$

$$= \frac{\cos^{-1}(1{\cdot}2799 \times 0{\cdot}4348)}{180°} \times 24$$

$$= \frac{\cos^{-1} 0{\cdot}5565}{180°} \times 24$$

$$= \frac{56{\cdot}2° \times 24}{180°} \text{(approx.)} = \text{7 hrs. 29 min.}$$

The following special cases are interesting:

(i) When $D = 0$.

(ii) The value of L for which $A = 0$ when $D = 23\frac{1}{2}$.

(iii) When $L = 0$.

(iv) $L < 66\frac{1}{2}°$ and $D = 23\frac{1}{2}°$.

The figure is drawn for North Latitude and winter. The duration of day for summer is obtained by subtracting the result found from the above formula when D is N. declination, from 24 hours. This is, of course, equal to the duration of night at corresponding winter date.

The time of sunrise and of sunset is easily determined, since the direction *ob* (plan) indicates noon. In the example worked, the sunrise is at $12 - \frac{7 \text{ h. } 29 \text{ m.}}{2} = 8$ h. 16 m. a.m.

EXERCISE XIX (B)

1. From the tables, find the cosecant, secant and cotangent of: 20°, 50°, 75°, 85°.

2. Resolve a force of 100 lbs. in directions making 60° and 30° on different sides of the direction of the force.

3. By means of a rope, a horse exerts a force of 200 lbs. upon a railway truck. If the rope makes an angle of 35° with the rails, calculate the force urging the truck along the rails when both rope and rails are horizontal.

4. A train is going E. at the rate of 80 ft. per sec. A rifle, held at right angles to the train, is discharged by a passenger, and it is found that the bullet follows a horizontal course 81° N. of E. Find the velocity of the bullet.

5. If the area ABCD (fig. 5) is 20 square inches, find the area of its projection on a plane making 50° with it.

6. A cylinder, 10 inches in diameter, is cut by a plane making 35° with its axis. Calculate the area of the section. Determine also the axes of the section, and check your first result by calculating the area from the lengths of the axes.

   (Area $=\pi$ times the product of the semi-axes.)

The angles of declination for the following problems will be found in *Whitaker's Almanack*:

7. Find the duration of day in London on May 14th, June 22nd, August 10th, September 23rd and January 31st.

   Determine also the approximate time of sunrise and sunset.

8. Determine the length of night at the tropic of Cancer on September 21st, December 22nd, March 1st and July 31st.

9. Determine the latitude beyond which there is perpetual night on (i) November 24th, (ii) January 31st, (iii) June 30th.

10. Find the horizontal range of a gun having a muzzle velocity of 2000 feet per second, the quadrant angle being 30°.

    What altitude will the projectile attain?

11. At what angle must a gun be set to hit a Zeppelin 6000 feet high at a horizontal distance of 1000 yards, the muzzle velocity being 2000 feet per second?

12. An aviator is flying at 60 miles per hour in a direction 25° N. of E., and a west wind is blowing 20 miles per hour. What is the direction of the wind relative to the aviator? (*N.B.*—A west wind blows to the east.)

    In what direction would the aeroplane travel in calm air?

13. Assuming the trajectory of a projectile to be a parabola, determine its equation when the maximum altitude is 2 miles and the horizontal range 12 miles.

14. Determine the equation to the parabolic trajectory which has a maximum altitude of 12,000 feet and a horizontal range of 15,000 feet.

15. Arrange equation (v) (page 221) in a convenient form for calculating V.

    Calculate V when $e=15°$ and $y=1$ when $x=4$ miles. Under what conditions would V be imaginary?

§9. The Trigonometrical Functions of an Angle and those of its double.

(i) $\sin 2A$.

Referring to the figure, ACB is a triangle in a semicircle. The angle C is therefore a right angle.

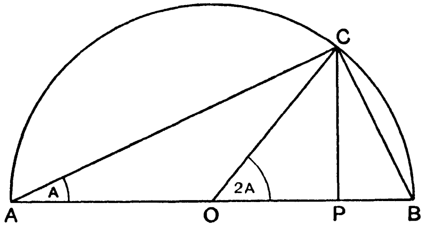


FIG. 8.

If the angle BAC is A, $\angle$BOC is 2A.

CP is perpendicular to AB.

From right-angled $\triangle$COP,

$$\sin 2A = \frac{CP}{OC} = \frac{CP}{OA} = \frac{CP}{\frac{AB}{2}}$$

$$= \frac{2CP}{AB}$$

$$= \frac{2CP}{AC} \cdot \frac{AC}{AB},$$

i.e. from right-angled $\triangle$s ACP and ACB,

$$\sin 2A = 2 \sin A \cos A. \quad \text{.......................(i)}$$

(ii) $\cos 2A$.

$$\cos 2A = \sqrt{1 - \sin^2(2A)}$$

$$= \sqrt{1 - 4\sin^2 A \cos^2 A}, \quad \text{from (i)},$$

$$= \sqrt{1 - 4\sin^2 A(1 - \sin^2 A)}$$

$$= \sqrt{1 - 4\sin^2 A + 4\sin^4 A}$$

$$= \sqrt{(1 - 2\sin^2 A)^2},$$

i.e. $\cos 2A = 1 - 2\sin^2 A$, ...(ii)

or, writing $1 - \cos^2 A$ for $\sin^2 A$,

$$\cos 2A = 1 - 2(1 - \cos^2 A),$$

i.e. $\cos 2A = 2\cos^2 A - 1$. ....(iii)

Also, since $1 - 2\sin^2 A = 1 - \sin^2 A - \sin^2 A$

and $1 - \sin^2 A = \cos^2 A$,

$$\cos 2A = \cos^2 A - \sin^2 A. \quad \text{.......................(iv)}$$

Relation (ii) can be established from the figure also.

(iii) tan 2A.

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}.$$

Dividing numerator and denominator by $\cos^2 A$

$$= \frac{\dfrac{2\sin A}{\cos A}}{1 - \dfrac{\sin^2 A}{\cos^2 A}},$$

$$\text{i.e.} \quad \tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \quad \text{.............................(v)}$$

These results are really particular forms of more general formulae, viz. :

$$\sin(A + B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B,$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

EXERCISE XIX (c).

1. Obtain equations (i), (ii) and (v) from the general formulae.
2. From equations (ii), (iii) and (v), show that

$$\sin \tfrac{1}{2}A = \sqrt{\frac{1 - \cos A}{2}}, \quad \cos \tfrac{1}{2}A = \sqrt{\frac{1 + \cos A}{2}}$$

and $\tan \frac{1}{2}A = \sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$.

3. From the trigonometrical ratios of 35°, find those of 70°.
4. Calculate sin, cos and tan $22\frac{1}{2}$°.
5. From the values of the trigonometrical ratios of 60° and 45°, find those of 105°.
6. For what value of a will
   (i) $\sin 2a$ be a maximum? (ii) $\sin 2a = 0$?
   (iii) $\cos 2a$ be a maximum? (iv) $\cos 2a = 0$?

7. Referring to fig. 8, page 225, show that the area of triangle ACB is $R^2 \sin 2A$.

   For what value of A is the area a maximum?

8. If $\sin A = x$, find $\sin 2A$ and $\cos 2A$ in terms of x. Find also $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ in terms of x.

9. Express $\sin^2 2x$ in terms of $\sin x$.

10. Find $\sin A$ in terms of $\sin \frac{1}{2}A$.

11. Simplify: $\tan(x+y) - \tan x$.

12. Show that $(\cos A + \sin A)^2 = 1 + \sin 2A$.

CHAPTER XX

FUNCTIONAL NOTATION, VARIATION, EXPANSION OF BINOMIALS, APPROXIMATIONS, FORMULAE

§ 1. Functional Notation.

A special notation is used to denote functions. The signs most generally used are, $f(x)$, $F(x)$ and $\phi(x)$, the letter in the brackets indicating the variable.

Thus, $f(x) = x^2 - 3x + 2$ means that the expression $x^2 - 3x + 2$ is a function of x. Similarly, $f(x) = ax^2 + bx + c$ means that the expression has to be regarded as a function of x only. That is, x is the only symbol which changes in value; the others being therefore constants.

The value of a function when a definite value, say 3, is given to the variable, is sometimes referred to as $f(3)$.

Thus, if $f(x) = x^2 - 3x + 2$, then $f(3) = 9 - 9 + 2 = 2$.

Change of Variable.

Example.—Represent $f(x) = x^2 - 3x + 2$ as a function of z, given that $z = x + 1$.

Since $z = x + 1$, $x = z - 1$.

Substitute this value of x in the given expression; then

$$f(z) = (z-1)^2 - 3(z-1) + 2,$$

$$\text{i.e.} \quad f(z) = z^2 - 5z + 6.$$

Compare this with the change in the position of the axis of Y (page 165).

Exercise XX (a)

1. If $f(x) = x^2 - 3x + 5$, find $f(2)$, $f(0)$ and $f(-3)$.

2. If $f(x) = 2x^2 - x + 1$, represent the expression as a function of z when $z = (x - 1)$.

3. If $f(x) = (2x + 1)(2x - 1) - (x + 1)$, find $f(y)$ when $y = (x - 1)$, and solve the equation $f(y) = 3$.

4. If $f(x) = \dfrac{(x+3)(6x-7)}{(3x+5)(2x-5)}$, find $f(-2)$, and state for what values of the variable x, $f(x) = 0$.

5. If d is a function of v such that
$$d = \frac{v^2 - u^2}{2a},$$
and v is a function of t such that
$$v = u + at,$$
express d as a function of t.

6. Express $\dfrac{3x + 11}{x^2 + 7x + 12}$ as the sum of two functions of x (see Exercise XII (F), No. 31).

7. Express $\dfrac{6x - 17}{6x^2 + 5x - 6}$ as the difference between two functions of x.

8. If $f(x) = \sin 2x \cos x$, find $\dfrac{f(30^\circ)}{f(45^\circ)}$.

9. If $f(x) = 3 \tan(x + 15^\circ)$, find $f(0^\circ)$ and $f(30^\circ)$.

§ 2. Variation.

Generally, one quantity is said to vary with another when one is so dependent upon the other that it changes when a change is made in the other. The law defining the change may be simple or may be complex.

Examples.

(1) The perimeter of a square depends on the length of the side of the square.

(2) The area of a circle depends on the length of the radius.

(3) The weight of a liquid depends upon the volume of the liquid, the temperature being constant.

(4) The weight of a bag of sovereigns depends upon the number of sovereigns the bag contains.

(5) The amount of interest due depends upon the principal invested, the time and the rate paid.

As to how these quantities vary is a matter for consideration. At present we need remark only that one quantity is a function of the other.

(1) *Direct Variation.* One quantity is said to vary directly as another when one is so dependent upon the other that the ratio of any two values of one quantity is equal to the ratio of the corresponding values of the other. Or if when one is changed, the other is changed in the same ratio.

E.g. the weight of water varies as the volume taken.

Thus: 10 cub. ft. weigh 625 lbs.,
6 cub. ft. weigh 375 lbs.,
2 cub. ft. weigh 125 lbs.

The ratio $\frac{10}{6}$ will be found equal to the ratio $\frac{625}{375}$,
also $\frac{6}{2}$ will be found equal to the ratio $\frac{375}{125}$,
and similarly for other corresponding values.

If V represents the volume of water and W the weight of this volume, then W varies as V. The sign $\propto$ stands for 'varies as,' and thus we write:
$$W \propto V.$$

Now, returning to our example, if we divide each weight by the corresponding volume, we obtain the same result in each case, thus: * $\frac{625}{10} = 62\cdot5$, $\frac{375}{6} = 62\cdot5$, $\frac{125}{2} = 62\cdot5$.

Looking at the reverse operation, we see that to get the numerical value of the weight from that of the volume, we must multiply the volume by 62·5,

$$\text{i.e. } W = 62\cdot5V.$$

The number 62·5 is called the constant, and is obtained by dividing one value of the quantity by the corresponding value of the other.

Hence, when we meet a statement like

$$W \propto V,$$

we can at once write it in the form of an equation, thus:

$$W = KV,$$

* **These fractions are not ratios in the strict sense of the word, because the terms are not of the same kind. They merely represent quotients.**

where K is a constant which can be obtained by dividing a value of W by the corresponding value of V.

For a more general proof, suppose V_1, V_2, V_3, etc., represent values of one quantity, and W_1, W_2, W_3, etc., represent corresponding values of the other; then, since W varies as V,

$$(1)\ \frac{V_1}{V_2}=\frac{W_1}{W_2}, \qquad (2)\ \frac{V_2}{V_3}=\frac{W_2}{W_3}, \qquad (3)\ \frac{V_1}{V_3}=\frac{W_1}{W_3}, \text{ etc.}$$

Transposing, we have that

$$\frac{W_2}{V_2}=\frac{W_1}{V_1}, \qquad \frac{W_3}{V_3}=\frac{W_2}{V_2}, \qquad \frac{W_3}{V_3}=\frac{W_1}{V_1}.^*$$

Examining these, it is evident that all these equations are equal to a constant number, which we have called K,

$$\text{i.e.}\quad \frac{W_1}{V_1}=\frac{W_2}{V_2}=\frac{W_3}{V_3}, \text{ etc.}=\text{K}.^*$$

Graph the numbers given as corresponding values of weight and volume, and draw your conclusions from the appearance of the graph.

You will find that the graph is a straight line of gradient K.

Hence W is a linear function of V.

(2) One quantity may vary as the inverse of another, or the square, cube, square root, etc., of another.

For example, in your Mensuration or Science lessons, you will find that the area of a circle varies as the square of the radius; that the volume of a sphere varies as the cube of the radius; that the time of the swing of a simple pendulum varies as the square root of the length; that the volume of a gas at constant temperatures varies inversely as the pressure. One quantity is said to vary *conjointly* as a number of others when it varies as their product. Thus, the value of a bar of gold varies conjointly as its length, breadth and thickness.

EXAMPLE.—A pendulum 100 cms. long takes 2 secs. to swing to and fro. Find the time for a pendulum of 36 cms. to swing to and fro.

Let t represent the time of swing, say in secs., and l the length, say in cms., of any pendulum; then t varies as $\sqrt{l}$;

$$\therefore\ t=\text{K}\sqrt{l} \quad\text{and}\quad \therefore\ \text{K}=\frac{t}{\sqrt{l}}.$$

* These fractions are not ratios in the strict sense of the word, because the terms are not of the same kind. They merely represent quotients.

To find K, make use of the given value of t (2 secs.) for the pendulum of length 100 cms.

$$K = \frac{2}{\sqrt{100}} = \frac{2}{10} = \cdot 2.$$

$\therefore$ the equation connecting t and l is

$$t = \cdot 2\sqrt{l}.$$

To find the time for a pendulum 36 cms. long,

$$t = \cdot 2\sqrt{36} = 1\cdot 2 \text{ secs.}$$

Exercise XX (b)

1. If $a \propto b$, and $b = 12$ when $a = 2$, find the equation connecting a and b, and find a when b is $4\cdot 5$.

2. If $x \propto y$ and $y \propto z$, show that $x \propto z$.

3. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

4. The weight of a cable of given thickness and material varies as the length. If a length, 120 yards, of this cable weighs 385 lbs., find the weight of a length 3 miles. Find also the weight per mile.

5. Show that the volumes of similar cones vary as the cube of their altitudes.

6. Show that the volumes of spheres vary as the cube of their radii.

7. The sag in a telegraph wire varies directly and conjointly as the length and the weight, and inversely as the horizontal tension.

   When the weight is 2 ozs. per foot, the length 80 feet and the horizontal tension 150 lbs., the sag is 8 inches. Find the sag when the weight is $1\frac{1}{2}$ ozs. per foot, the length 20 yards and the tension 100 lbs.

8. If $y^n = \frac{k}{x}$, find n and k, given that x is 5 when y is 10, and is 11 when y is 8.

9. The period of a planet, that is, the time it takes to make one revolution round the sun, is found to vary as the square root of the cube of its distance from the sun. Knowing the period and distance of the earth, find the distance of Jupiter, the period of which is observed to be $11\cdot 86$ years.

10. In printing gas-light photographic papers, the time of exposure varies as the square of the distance of the plate from the source of light. If for a distance 8 inches the time is 6 seconds, what exposure is necessary for a distance 18 inches? For what distance would the time be 12 seconds?

§ 3. Rapid Expansions.*

We have seen that

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3,$$
$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4.$$

Notice that the terms run in descending powers of a and ascending powers of b, and that when the sign between the two given terms is + the signs of the expansion are all +, and that when −, the signs of the expansion run alternately + and −, the first sign being +, although not shown.

Look at the last example, and verify this rule for finding the coefficient of a term, say the third term, from the preceding term. Examine the second term. Multiply its coefficient (4) by the index of the descending power, i.e. the index 3 of a^3, and divide the product by the number of the term, i.e. being the second term by 2. The result is the coefficient of the next term. Thus, $\frac{4 \times 3}{2} = 6$.

Try the second term the same way. Remember the coefficient of the first term is 1.

To expand $(a - b)^5$.

First write down the terms without their coefficients in descending powers of a and ascending powers of b.

Thus, $(a-b)^5$, $a^5 \quad - \quad a^4b \quad + \quad a^3b^2 \quad - \quad a^2b^3 \quad + \quad ab^4 \quad - \quad b^5$.
1st 2nd 3rd 4th 5th 6th

Then calculate the coefficients

$$\frac{1 \times 5}{1} \searrow \frac{5 \times 4}{2} \searrow \frac{10 \times 3}{3} \searrow \frac{10 \times 2}{4} \searrow \frac{5 \times 1}{5} \searrow$$
$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$$

The expansion can be readily checked by putting $a = 1$ and $b = 1$; then each side should equal 0.

* The general formula for the expansion of a binomial is given in Chapter XXV.

The coefficients of the expanded powers of $(a+b)$ can be arranged so as to show how one set of coefficients can be obtained from those of the next lower power. Thus:

	Coefficients in order.
$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
etc.	

The first and last coefficients are always unity. The brackets indicate that the coefficients of one expanded power when added in successive pairs give the intermediate coefficients of the next power.

The arrangement suggests a triangle, and is known as Pascal's triangle—Pascal being the name of the discoverer.

The reason underlying Pascal's triangle is readily understood if two successive powers are compared.

Take, for example, $(a+b)^4$ and $(a+b)^5$.

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

$$\begin{aligned}
(a+b)^5 &= (a+b)(a+b)^4 \\
&= (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
&= b(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
&\qquad\qquad + a(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
&= a^5 + (1+4)a^4b + (4+6)a^3b^2 + (6+4)a^2b^3 \\
&\qquad\qquad + (4+1)ab^4 + b^5.
\end{aligned}$$

This arrangement shows how the coefficients of the expansion of $(a+b)^5$ are obtained from those of the expansion of $(a+b)^4$.

Exercise XX (c)

1. Expand $(x+y)^4$ and $(x+y)^6$.
2. Expand $(x-y)^4$ and $(x-y)^6$.
3. Expand $(a-b)^3$, $(a+b)^7$, $(a-b)^5$.
4. Expand $(a^2-b^2)^5$.

5. Expand $(2a-3b)^4$. (Take $(x-y)^4$ as a pattern, and substitute in the expansion, $2a$ for x and $3b$ for y.)

6. Find $(2a-3b)^7$. 7. Expand $\{(a+b)+(b+c)\}^3$.

8. Find $\{(a+b)-(c+2)\}^4$.

9. Find the fifth power of $(x^2+y^2-z^2)$.

10. In the expansion of $(a+b)^5$, put a and b each equal to 1, and then find the sum of the coefficients.

11. Find, correct to the second place of decimals, $(1{\cdot}005)^8$.

12. Expand $(1-2a)^{10}$.

§ **4. Approximations.**

A small quantity, say a small increase or decrease, is usually denoted by the symbol δ or δx. The latter does not mean δ times x, but is equivalent to a single symbol.

Consider the expansion $(1+x)^2=1+2x+x^2$.

If x is small compared with unity, i.e. if x is a small fraction, x^2, being a fraction of a fraction, will be smaller still. Thus, if x is 0·1, then x^2 is 0·01. Generally, the square, and therefore the higher powers of very small numbers, may be neglected.

If we write δ for x, we have

$$(1+\delta)^2=1+2\delta \text{ approximately.}$$

Similarly, if n represents any power,

$$(1+\delta)^n=1+n\delta.$$

Examples.

$$(1{\cdot}0012)^2=(1+{\cdot}0012)^2=1+{\cdot}0012\times 2$$
$$=1{\cdot}0024 \text{ approx.,}$$
$$(1{\cdot}0006)^7=1+{\cdot}0006\times 7=1{\cdot}0042 \text{ approx.}$$

The same is true for roots. Thus:

$$\sqrt{1+\delta}=(1+\delta)^{\frac{1}{2}}=1+\tfrac{1}{2}\delta.$$

Example. $$\sqrt[3]{1{\cdot}0026}=1+\frac{{\cdot}0026}{3}=1{\cdot}00087 \text{ approx.,}$$

$$\sqrt[5]{0{\cdot}9995}=(1-{\cdot}0005)^{\frac{1}{5}}=1-\frac{{\cdot}0005}{5}$$
$$=1-{\cdot}0001$$
$$={\cdot}9999 \text{ approx.}$$

Applications.

1. *Expansion.*

(i) *Area.* A metal plate has the shape of a square, and its edge is of unit length. When its temperature is raised one degree, each edge increases in length by a small amount δ, called the coefficient of linear expansion. Find the increase in area.

Area before the temperature is raised $= 1$ unit of area.

" after " " " $1^\circ = (1+\delta)^2$ units of area.

The increase in area for 1° rise in temperature $= (1+\delta)^2 - 1$

$$= 2\delta + \delta^2.$$

Now δ is very small, e.g. for iron, per $^\circ$C., $\delta = {\cdot}0000117$; for copper, ${\cdot}000017$; and therefore δ^2 can be neglected. *The small corner square of the figure shows that δ^2 is small compared with 2δ.*

It follows that the coefficient of surface expansion is approximately 2δ, i.e. approximately twice the coefficient of linear expansion.

(ii) *Volume.* A metal cube of unit edge, when raised one degree in temperature, has each edge increased in length by a small amount δ. Find the increase in volume.

Volume before temperature raised $= 1$ unit of volume.

" after " " $1^\circ = (1+\delta)^3$ units of volume.

Increase in unit volume for 1° rise $= (1+\delta)^3 - 1$ " "

$$= 3\delta + 3\delta^2 + \delta^3 \quad " \quad "$$

δ being small, the terms $3\delta^2$ and δ^3 are negligible (fig. 1).

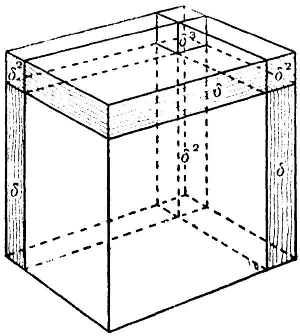


FIG. 1.

It follows that the coefficient of volume expansion is 3δ, i.e. three times the coefficient of linear expansion.

For $t°$ rise in temperature, the surface expansion is $2\delta t$ per unit area, and the volume expansion $3\delta t$ per unit volume.

2. The force (F) of a magnet at a point a distance x from the middle point of the straight line joining the poles, and in that straight line produced, is given by the equation:

$$F = \frac{m}{(x-d)^2} - \frac{m}{(x+d)^2}, \quad \text{.......................(i)}$$

where m is the pole strength and d is half the distance between the poles. Both m and d are constant for a particular magnet.

Show that the equation reduces to:

$$F = \frac{4mxd}{(x^2-d^2)^2}. \quad \text{.......................(ii)}$$

The denominator $(x^2-d^2)^2$ can be arranged in the form:

$$\left\{x^2\left(1 - \frac{d^2}{x^2}\right)\right\}^2.$$

Look carefully at the term $\frac{d^2}{x^2}$, which is equal to $\left(\frac{d}{x}\right)^2$.

As x is made greater, $\frac{d}{x}$ becomes smaller, and $\left(\frac{d}{x}\right)^2$ smaller still, since $\frac{d}{x}$ is a fraction.

Hence, when x is great compared with d, the term $\frac{d^2}{x^2}$ is generally small enough to be neglected.

In such a case the equation:

$$F = \frac{4mxd}{\left\{x^2\left(1 - \frac{d^2}{x^2}\right)\right\}^2}$$

becomes
$$F = \frac{4mxd}{(x^2)^2}$$

which reduces to
$$F = \frac{4md}{x^3},$$

i.e. when x is great compared with d, the force of the magnet varies inversely as the cube of x.

Exercise XX (d)

1. The dimensions of a rectangular plate of copper at 0° C. are 18″ by 15″. Find the area of its surface at 50° C.

2. A cylinder of copper has a diameter of 12 cms. at 15° C. Find the area of one end at 100° C. If its length at 15° C. is 15 cms., find its volume at 100° C.

3. The volume of a flask is 250 c.c. at 0° C. What will be its volume at 100° C.? (Coefficient of linear expansion of glass ·000009.)

4. The diameter of a spherical glass bulb is increased by 1 per cent. By what percentage is its capacity increased?

5. The capacity of a glass flask at 0° C. is 1 litre. If it contains air, what volume will escape when the flask is heated to 100° C.? (Coefficient of expansion of air is $\frac{1}{273}$ per degree C.)

6. Find approximately:

$$\sqrt[5]{1\cdot 0003}, \qquad \sqrt[3]{0\cdot 997}, \qquad \sqrt[3]{1002\cdot 1},$$
$$(1\cdot 002)^2, \qquad (\cdot 996)^4, \qquad (10\cdot 06)^2.$$

7. By how much must the temperature of a sheet of iron be raised in order that its surface area may be increased by 1 per cent.?

8. At what temperature will a rod of copper 199 cms. long at 15° C. and a rod of iron 200 cms. long at 15° C. have the same length?

9. If δ is a small fraction, show that $\frac{1}{1+\delta} = 1 - \delta$ approximately.

10. The distance between the poles of a magnet is 4 cms. Find the percentage error made in using the formula $\frac{4md}{x^3}$ to find the force at a point 20 cms. from the middle point of the line joining its poles, and in this line produced.

11. Show that the formula $\frac{m}{x-d} - \frac{m}{x+d}$ reduces to $\frac{2md}{x^2}$, when x is great compared with d.

12. Find the error per cent. when using the second formula in Question 11 instead of the first, when $d = 4$ and $x = 20$.

§ 5. Particular and General Formulae.

In many cases a given formula is a particular example of a more general formula.

For example, the formulae for the circumference and area of a circle are particular cases of the general formulae for the ellipse.

If a and b are the semi-axes of an ellipse, then

(i) the circumference $= \pi(a+b)$, *approximately, the formula being more exact the nearer* a *is equal to* b;

and (ii) the area $= \pi ab$.

In the circle, the semi-axes are equal, i.e. $a = b$, and substituting r for each, we have:

(i) Circumference of circle $= \pi(r+r) = 2\pi r$.

(ii) Area of circle $= \pi r^2$.

Exercise XX (e)

1. The area of a sector of a circle is $\frac{1}{2}\text{R}a$, where a = the length of the arc and R the radius. What does this formula become when the sector is a semicircle? What, therefore, is the area of the whole circle?

2. The area of the curved surface of a spherical cap is $2\pi\text{R}t$, where R is the radius of the sphere and t the altitude of the cap. What does this become when the cap is a hemisphere? What then is the surface of the whole sphere?

3. The volume of an ellipsoid is $\frac{4}{3}\pi abc$, where a, b and c are its semi axes. What does this become for the sphere, in which the semi axes are equal?

4. The volume of the cap of a sphere is $\frac{1}{6}\pi t(t^2+3r^2)$, where t is the altitude and r the radius of the base of the cap. Apply this formula to a hemisphere, and to a whole sphere.

5. The area of a segment of a circle is given approximately by the formula $\dfrac{h}{6b}(4b^2+3h^2)$, where h is the altitude and b the base of the segment. Apply this to the semicircle, and find the error per cent.

6. Another formula for the area of a segment of a circle is $\frac{2}{3}h(b+\frac{4}{3}c)$, in which c is the chord of the semi arc. Apply this to the semicircle, and find the error per cent.

7. The curved surface of the frustum of a cone is $\pi s(\text{R}+r)$, where s is the slant height, and R and r the radii of the base and top. What does this become for the full cone?

8. The volume of the frustum of a cone is $\frac{1}{3}\pi h(\text{R}^2+\text{R}r+r^2)$, where h is the altitude of the frustum. What does this become for the full cone? Apply the given formula to a cylinder.

CHAPTER XXI

PROGRESSIONS, SERIES

§ 1. Progressions.

By a progression we mean a series of numbers which proceeds in order according to some law.

The two simpler progressions are *Arithmetical Progression* and *Geometrical Progression.*

(i) *Arithmetical Progression* (A.P.).

In an Arithmetical Progression, the terms proceed by equal added amounts (or differences).

The added amount may be positive or negative.

EXAMPLES.

(i) 2, 5, 8, 11, 14, 17, etc.

The above numbers form an Arithmetical Progression.

Commencing from the first number, 2, the second, 5, is obtained by adding 3 to the first. Similarly, the third number, 8, is obtained by adding 3 to the second, and so on.

The added amount, or *common difference* as it is more usually called, is found by taking any term and subtracting from it the preceding term.

(ii) 10, 5, 0, -5, -10, etc.

The common difference in this A.P. is -5. Verify this statement.

(iii) $3a$, $7a$, $11a$, $15a$, etc.

This is an A.P. having a common difference $4a$. Verify this.

EXERCISE XXI (A)

1. Write down three more terms to each of the given examples.
2. Write down a few terms of the A.P. of which the first term is $3a$ and the common difference $-4a$. Contrast this progression with Example iii.
3. Write eight terms of the following Arithmetical Progressions:

(i)	First	term	6,	common	difference	4.
(ii)	„	„	6	„	„	-4.
(iii)	„	„	-6	„	„	4.
(iv)	„	„	-6	„	„	-4.
(v)	„	„	1	„	„	-4.
(vi)	„	„	0	„	„	-2.

4. Construct the A.P. of which the first term is a and the common difference d. Compare the coefficient of d in any term with the number of that term.

(ii) *Geometrical Progression* (G.P.).

In a Geometric Progression, the terms proceed by a constant ratio. In other words, the ratio any term bears to the preceding term is the same throughout the series.

The constant ratio may be positive or negative.

EXAMPLES.

(i) 2, 6, 18, 54, 162, etc.

The above numbers form a Geometrical Progression.

Commencing from the first number, 2, the second, 6, is obtained by multiplying the first by 3. Similarly, the third, 18, is obtained by multiplying the second by 3, and so on.

The common ratio is found by dividing any term by the preceding term.

Contrast this progression with the A.P., Example (i), page 239.

(ii) $12, \ -3, \ \frac{3}{4}, \ -\frac{3}{16}$, etc.

The common ratio of this G.P. is $\frac{-3}{12} = -\frac{1}{4}$. Verify this statement.

(iii) $3a, \ 6a^2, \ 12a^3, \ 24a^4$, etc.

This is a G.P. having a common ratio $2a$. Verify this.

EXERCISE XXI (B)

1. Extend each of the series given by three terms.

2. Write down a few terms of the G.P. of which the first term is $3a$ and the common ratio $-2a$. Contrast this progression with Example iii.

3. Write down six terms of the following Geometrical Progressions:

(i)	First term	1,	common ratio		-2.
(ii)	,, ,,	1	,,	,,	$\frac{1}{2}$.
(iii)	,, ,,	-2	,,	,,	$-\frac{1}{3}$.

4. Construct the Geometrical Progression of which the first term is a and the common ratio r. Compare the index of the power of r of any term with the number (in order) of that term.

5. Determine whether the following series are in Geometrical or Arithmetical Progression. Give reasons in each case.

(i) 3, 6, 9, 12, etc. (ii) 3, − 6, 12, − 24, etc.
(iii) − 4, − 2, 0, etc. (iv) − 4, − 2, − 1, etc.

§ 2. Graphical Representation.

(i) *Arithmetical Progression.*

If the terms of an A.P. are plotted against the order of the terms, then, since they proceed by equal added amounts, the plotted points lie in a straight line (see page 96).

This straight line * has an up gradient if the added quantity is positive and a down gradient if negative.

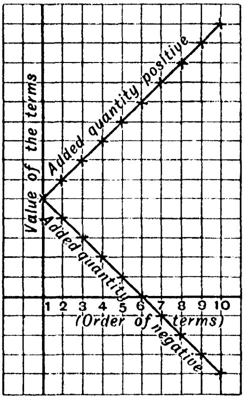


FIG. 1.

(ii) *Geometrical Progression.*

If we plot the terms of a G.P. against the order of the terms, the points lie on a curve.

CASE 1.—When the common ratio is positive and greater than unity, the terms increase and the curve diverges from the axis of x (fig. 2).

* *If the line is drawn it is not a graph in the sense in which we have already used the name, since the portions between the plotted points have no significance. The line serves to show only the relative position of the plotted points.*

CASE 2.—When the common ratio is positive and less than unity (fractional), the successive terms decrease and the curve gradually approaches the axis of x (fig. 3).

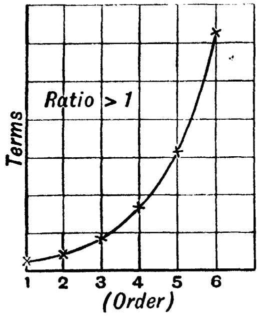


FIG. 2.

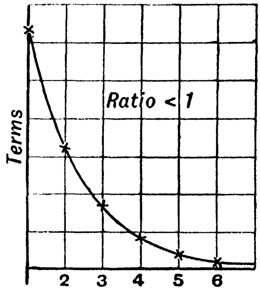


FIG. 3.

CASE 3.—When the common ratio is negative the terms are alternately positive and negative. Such a series can be regarded as consisting of two series, one positive and the other negative.

The common ratio of each set is the square of the ratio of the series. Prove this.

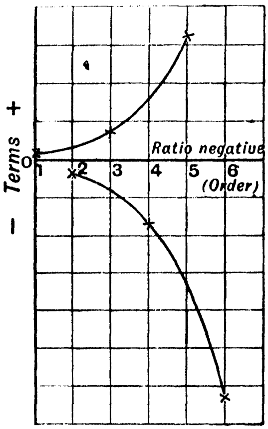


FIG. 4.

EXERCISE.—Apply the graphical method to Ex. XXI (B), No. 5.

§ 3. Arithmetical Progression.

(i) *General Term.*

In Exercise XXI (A), No. 4, you should have found that in the A.P., the first term of which is a and the common difference d, the coefficient of d is one less than the number of the term.

If we call any term the nth term, n representing the number of the term in order of succession, then

$$\text{The } n\text{th term is } \{a+(n-1)d\}.$$

This is called the general term, or the general expression for any term. From it any term can be found without writing down the terms which precede it.

EXAMPLE.—Find the 35th term of the series:

$$1, \quad 5, \quad 9, \quad 15, \quad \text{etc.}$$

Since the series proceeds by equal added amounts, it is an A.P.

The first term is 1 and the common difference 4.

The 35th term is $1+(35-1)4=1+34\times4=137$.

(ii) *Means.*

The terms between any two chosen terms of a series are called **Means.** In an Arithmetical Progression, such terms are called Arithmetic Means, and when there is only one term between the chosen terms it is called the Arithmetic Mean of the other two.

To insert a given number of Arithmetic Means (A.M.'s) *between given numbers.*

EXAMPLE.—Insert 6 A.M.'s between 2 and -26.

Let d = the common difference.

Then the terms are:

$$2, \quad (2+d), \quad (2+2d), \quad (2+3d), \quad (2+4d), \quad (2+5d),$$
$$(2+6d) \quad \text{and} \quad (2+7d) \text{ or } -26.$$

It is seen that -26 is the 8th term.

Hence
$$2+7d=-26,$$
$$7d=-28,$$
$$d=-4.$$

The means are therefore:

$$-2, \quad -6, \quad -10, \quad -14, \quad -18 \quad \text{and} \quad -22.$$

The *Arithmetic Mean* (A.M.) of two numbers is half their sum.

Let the numbers be a and b and their A.M., M.

Then a, M and b form an A.P.

Hence
$$M - a = b - M,$$
$$2M = a + b,$$
$$M = \frac{a+b}{2}.$$

(iii) *The Sum of a Number of Terms in* A.P.

Consider first a numerical series, say 2, 5, 8, 11, 14, 17, etc., and let us find the sum of, say, 7 terms.

Represent the sum by S; then

$$\left.\begin{array}{r} \\ \text{and reversing} \\ \text{the terms,} \end{array}\right\} \begin{array}{l} S = 2 + 5 + 8 + 11 + 14 + 17 + 20 \\ S = 20 + 17 + 14 + 11 + 8 + 5 + 2 \end{array}$$

By adding,
$$2S = 22 + 22 + 22 + 22 + 22 + 22 + 22$$
$$= 22 \times 7;$$
$$S = \frac{22 \times 7}{2},$$
i.e. $S = 77.$

Observe that the sum is half the product of the sum of the first and last terms and the number of terms,

i.e. $S = \dfrac{(\text{first} + \text{last}) \times \text{number of terms}}{2}.$

The General Formula.

Let the series be:

$$a, \quad (a+d), \quad (a+2d), \ \ldots \ (a+\overline{n-1}d);$$

then, if S_n represents the sum of n terms,

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+\overline{n-1}d)$$

and
$$S_n = (a+\overline{n-1}d) + (a+\overline{n-2}d) + (a+\overline{n-3}d) + \ldots + a$$

Adding,
$$2S_n = \underbrace{(2a+\overline{n-1}d) + (2a+\overline{n-1}d) + (2a+\overline{n-1}d) + \ldots + (2a+\overline{n-1}d)}_{n \text{ terms all alike}}$$
$$= n(2a+\overline{n-1}d);$$
$$S_n = \frac{n}{2}(2a+\overline{n-1}d).$$

As before, the result may be stated in the form:

$$S_n = \frac{n}{2}(\text{first term} + \text{last term}).$$

Exercise XXI (c)

1. Find the 21st term of the series: $1, \ -3, \ -7, \ -11$, etc.
2. The first term of a series is 2, and the tenth 29. Write down the first six terms.
3. Show that the series formed by adding each term of an A.P. to the succeeding term is an A.P.
4. Insert three arithmetic means between 3 and 18.
5. Insert four arithmetic means between 5 and -10.
6. Find the sum of the first 25 whole numbers.
7. Establish a formula for the sum of the first n integers.
8. Find the sum of the first 20 odd numbers.
9. Find the sum of the first 20 even numbers.
10. Establish formulae for the first n odd, and for the first n even numbers.
11. The number of dominoes required for a set is the sum of the following series:

$$1, 2, 3 \ldots x, (x+1),$$

where x denotes the highest number used (i.e. the highest domino is double x).

Find the number of dominoes required for a set, the highest domino of which is double six.

12. Find the sum of 20 terms of the series whose nth term is $(6n-2)$.
13. Plot the terms of an A.P., and on the same figure show a rectangle, the area of which represents the sum of the terms.
14. If a, b, c and d are consecutive terms of an A.P., show that $bc - ad = 2(b-c)^2$.

§ 4. Geometrical Progression.

(i) *General Term.*

Your answer to Exercise XXI (B), No. 4, should be that the index of r is always one less than the number of the term.

Thus the nth term of the G.P. a, ar, ar^2, ar^3, etc., is ar^{n-1}.

This enables us to write down any term without determining the preceding terms.

EXAMPLE.—Find the tenth term of the series:

$$3, \quad -6, \quad 12, \quad -24, \quad \text{etc.}$$

Here a is 3 and r is $-6/3$, i.e. -2.

Hence the tenth term is $3 \times (-2)^9 = 3 \times -512 = -1536$.

If logarithms are used in such exercises as the above, it must be remembered that the accuracy of the result depends upon the range of the logarithms used.

Applying logarithms to the formula for the general term, we have:

$$\log(n\text{th term}) = \log a + (n-1)\log r.$$

(ii) *Geometric Means.*

The terms between any two terms of a Geometrical Progression are called Geometric Means.

When there are three terms only, the middle term is called the Geometric Mean (G.M.) of the other two.

To insert a given number of Geometric Means between given numbers.

EXAMPLE.—To insert three Geometric Means between 2 and 162.

Let r be the common ratio; then the terms are:

$$2, \quad 2r, \quad 2r^2, \quad 2r^3 \quad \text{and} \quad 2r^4, \text{ or } 162.$$

Hence
$$2r^4 = 162,$$
$$r^4 = 81,$$
$$r = \pm 3.$$

The means are, therefore, $6, \ 18, \ 54,$

or $-6, \ 18, \ -54.$

The *Geometric Mean* of two numbers is the square root of their product.

Let G represent the G.M. of a and b.

Then a, G and b are in G.P.

Hence
$$\frac{G}{a} = \frac{b}{G},$$
$$G^2 = ab,$$
$$G = \sqrt{ab}.$$

Contrast this with the A.M. of a and b.

(iii) *Geometrical Constructions.*

1. *To find the G.M. of two given straight lines.*

Let the lengths of the lines be x and y units respectively.

Referring to the figure, AB $= x$ and BC $= y$, AC $= (x+y)$. Bisect AC at D; then DC $= \frac{1}{2}(x+y)$ and DB $= \frac{1}{2}(x-y)$. With centre D and radius DC, describe a semicircle on AC. From B, erect a perpendicular to meet the curve at P.

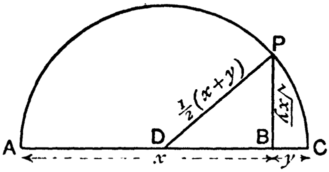


FIG. 5.

Join D and P.

Then $\quad$ DP $= \frac{1}{2}(x+y)$.

Now $\quad$ PB^2 = AB × BC $= xy$ (see Ch. VIII. § 5, 3).

Hence $\quad$ PB $= \sqrt{xy}$.

Observe that DP is the A.M. of x and y. It follows, since the hypotenuse of a right-angled triangle is its greatest side, that in general the A.M. of two numbers is greater than the G.M. Under what conditions are the two means equal?

2. *Given two straight lines, a and b, to find a third straight line c, such that a, b and c are in* G.P.

Place the lines a and b in a parallel position, as shown by AB and CD in fig. 6.

Join AC and BD, and produce these joining lines.

Join BC, and through D draw DE parallel to BC, to meet AC produced in E. From E draw EF parallel to CD, to meet BD produced, in F.

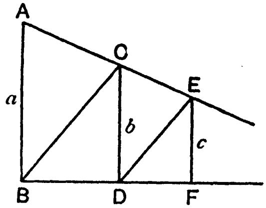


FIG. 6.

Then EF is the straight line required.

Since triangles ACB and CED are similar, as are also triangles CDB and EFD,

$$\frac{a}{b} = \frac{BC}{DE} \quad \text{and} \quad \frac{BC}{DE} = \frac{b}{c};$$

$$\therefore \frac{a}{b} = \frac{b}{c}.$$

Hence a, b and c are in G.P.

(iv) *The Sum of a Number of Terms in* G.P.

Let the terms be a, ar, ar^2, ar^3, etc., and let the sum of n terms be represented by S_n.

$$\begin{array}{lrl} \text{Then} & S_n = & a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ \text{Multiply by } r, & rS_n = & \phantom{a +{}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\ \hline \text{Subtract,} & S_n - rS_n = & a \ldots\ldots\ldots\ldots\ldots\ldots\ldots - ar^n \end{array}$$

$$S_n(1 - r) = a(1 - r^n),$$

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

When r is greater than 1, the above formula is better written in the form :

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

Hence, to find the sum, extend the series by one term, and divide the difference between this term and the first term by the difference between the common ratio and unity.

EXAMPLE —Find the sum of five terms of the series :

$$6, \quad -2, \quad \tfrac{2}{3}, \quad -\tfrac{2}{9}, \quad \text{etc.}$$

The numbers are in G.P.; the common ratio is $\dfrac{-2}{6} = -\dfrac{1}{3}$.

$$S_5 = \frac{6\{1 - (-\frac{1}{3})^5\}}{1 - (-\frac{1}{3})} = \frac{6(1 + \frac{1}{243})}{1\frac{1}{3}}$$

$$= 4\tfrac{14}{27}.$$

Verify this result by Arithmetic.

The following graphical representation of the sum is interesting:

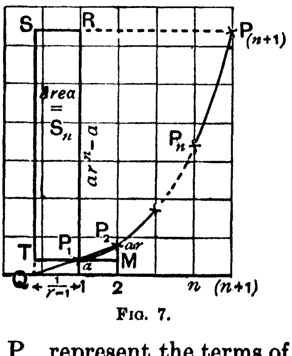


FIG. 7.

Let P_1, P_2, P_3, ... P_n, represent the terms of a G.P. (fig. 7); then

$$1P_1 = a, \quad 2P_2 = ar, \quad 3P_3 = ar^2 \ldots nP_n = ar^{n-1}.$$

Join P_2, P_1, and produce the straight line P_2P_1 to cut the axis of n at Q.

Draw P_1M parallel to Qn; then $P_1M = 1$ and $MP_2 = ar - a$, or $a(r-1)$.

Now triangle P_1Q1 is similar to triangle P_2P_1M; therefore

$$\frac{1Q}{1P_1} = \frac{P_1M}{MP_2},$$

$$\begin{aligned} 1Q &= 1P_1 \times \frac{P_1M}{MP_2} \\ &= a \times \frac{1}{a(r-1)} \\ &= \frac{1}{r-1}. \end{aligned}$$

The sum of n terms is $\dfrac{ar^n - a}{r-1}$ or $(ar^n - a) \times \dfrac{1}{r-1}$.

Draw QS parallel to 1R.

Take the next term, namely, the $(n+1)$th term. Its value is ar^n. Let P_{n+1} denote its position on the graph. Draw $P_{n+1}RS$ and P_1T parallel to Qn, and thus obtain the rectangle P_1RST. Then $P_1R = ar^n - a$ and $P_1T = \dfrac{1}{r-1}$, and therefore the area of P_1RST represents the sum of n terms.

The significance of the rectangle P_1TQ1 will be seen immediately.

The same construction holds good for a G.P., the terms of which are decreasing.

Fig. 8 illustrates such a series.

The common ratio is, of course, less than unity.

The area P_1RST represents the sum of n terms.

As the number of terms increases, the last term approaches more and more the value 0, the graph gets closer and closer to the axis of n, and the line RS gets nearer and nearer to 1Q, the rectangle P_1RST becoming more nearly equal to the rectangle P_11QT. When the number of terms is infinite, the rectangle representing their sum differs by no measurable amount from the rectangle P_11QT.

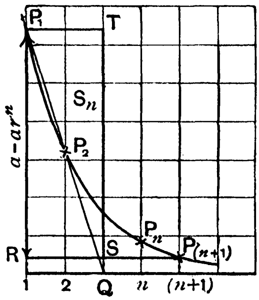


Fig. 8.

Hence $P_1 1QT$ represents the sum of an infinite number of terms of a G.P., the first term of which is a and the common ratio r, r being less than unity.

Now

$$\text{Area } P_1 1QT = 1P_1 \times 1Q$$
$$= a \times \frac{1}{1-r}$$
$$= \frac{a}{1-r},$$
$$\text{i.e.} \quad S_\infty = \frac{a}{1-r}.$$

Referring to fig. 7, the rectangle $P_1 1QT$ represents the sum of an infinite number of terms in G.P., the greatest of which is $1P_1$.

The result, $S_\infty = \frac{a}{1-r}$, may be deduced from the relation, $S_n = \frac{a - ar^n}{1-r}$.

When r is less than unity, r^n becomes smaller and smaller as n increases.

The term ar^n can be made to differ from 0 by as small a quantity as we please, by making n large enough.

To make ar^n actually 0, n must be infinite.

In this case, $S_\infty = \frac{a-0}{1-r} = \frac{a}{1-r}$.

EXAMPLE.—Find the sum of the series, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., without limit. Here $r = \frac{1}{2}$, and $S_\infty = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.

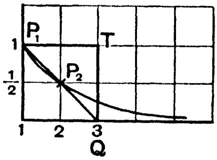


FIG. 9.

Fig. 9 shows how the result can be obtained graphically.

Recurring Decimals.

A recurring decimal is an example of a G.P. with an infinite number of decreasing terms.

Thus: $\cdot\dot{3} = \cdot 333333333333\ldots$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \text{etc.}$$

The common ratio is $\frac{1}{10}$, and the sum of the terms is the value of the given decimal.

$$S_\infty = \frac{a}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{3}{10} \times \frac{10}{9} = \frac{3}{9} \quad \text{or} \quad \frac{1}{3},$$

$$\text{i.e.} \quad \cdot\dot{3} = \frac{3}{9} \quad \text{or} \quad \frac{1}{3}.$$

Similarly,

$$\cdot 2\dot{3} = \frac{2}{10} + \left[\frac{3}{100} + \frac{3}{1000} + \dots \text{etc.}\right]$$

$$= \frac{2}{10} + \frac{\frac{3}{100}}{1-\frac{1}{10}}$$

$$= \frac{2}{10} + \frac{3}{90}$$

$$= \frac{18+3}{90} \quad \text{or} \quad {}^{*}\frac{2(10-1)+3}{90} = \frac{20-2+3}{90}$$

$$= \frac{21}{90} \qquad\qquad = \frac{23-2}{90}$$

$$= \frac{7}{30}.$$

Exercise XXI (d)

1. Find the 8th term of the series 2, 6, 18, 54, etc.
2. Find the 9th term of the series -2, 6, -18, 54, etc.
3. Show that the series obtained by adding each term of a G.P. to the succeeding term is a G.P.
4. Show that the series obtained by subtracting each term of a G.P. from the succeeding term is a G.P.
5. Show that the series obtained by multiplying each term of a G.P. by the succeeding term is a G.P.
6. Insert three G.M.'s between -3 and -768.
7. Find a straight line c, such that the side of the square of area 2·25 sq. inches, the diagonal and c are in G.P.
8. Find the sum of the first six terms of the series 2, 6, 18, 54, etc., and of the series 2, -6, 18, -54, etc.
9. Find the sum of eight terms of the series 6, 3, $1\frac{1}{2}$, $\frac{3}{4}$, etc.
10. Find the sum of $\frac{3}{4}$, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{2}{9}$ ... to 10 terms.

*The alternative steps explain the rule for converting a recurring decimal into a vulgar fraction.

11. The fourth term of a G.P. is 24, and the ninth term -768. Find the eleventh term.

12. If s is the sum of a G.P., in which the first term is a and the last b, show that the common ratio is $\dfrac{s-a}{s-b}$.

13. Express as vulgar fractions, $\cdot 5\dot{2}\dot{4}$ and $\cdot 2\dot{3}0\dot{7}$, from first principles.

14. Find the sum of an infinite number of terms of the series $1-\delta+\delta^2-\delta^3+\text{etc.}$, when δ represents a fraction.

15. Draw a straight line 2 inches long, and by successive bisection illustrate that the sum of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, etc., without limit, is 2.

§ 5. Harmonical Progression.

1. Numbers are in harmonical progression when their reciprocals are in arithmetical progression.

EXAMPLES.

(i) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, etc., are in H.P., because their reciprocals, 2, 3, 4, 5, etc., are in A.P.

(ii) $\frac{1}{2}, \frac{4}{7}, \frac{2}{3}, \frac{4}{5}, 1, 1\frac{1}{3}, 2$, etc., are in H.P., because their reciprocals, $2, 1\frac{3}{4}, 1\frac{1}{2}, 1\frac{1}{4}, 1, \frac{3}{4}, \frac{1}{2}$, etc., are in A.P.

(iii) Since the general form of the A.P. is

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+3d, \ldots a+\overline{n-1}d,$$

the general form of the H.P. is

$$\frac{1}{a}, \quad \frac{1}{a+d}, \quad \frac{1}{a+2d}, \quad \frac{1}{a+3d}, \cdots \frac{1}{a+\overline{n-1}d}.$$

2. If a, b and c are in H.P., then $\dfrac{1}{a}$, $\dfrac{1}{b}$ and $\dfrac{1}{c}$ are in A.P.

Hence $$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},$$

from which $$\frac{a-b}{ab}=\frac{b-c}{bc}$$

and $$\frac{a-b}{b-c}=\frac{a}{c},$$

i.e. the ratio of the excess of the first over the second to the excess of the second over the third, is equal to the ratio of the first to the third.

Harmonical progression is often defined in this way. The definition given in 1 is more easily remembered.

3. Problems on harmonical progressions are most conveniently solved by transforming the series by inversion into the corresponding arithmetical progression.

EXAMPLE.—Find the harmonic mean (H.M.) between a and b.

Let x be the H.M.

Then $\frac{1}{a}$, $\frac{1}{x}$ and $\frac{1}{b}$ are in A.P.

Hence
$$\frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x},$$
$$\frac{2}{x}=\frac{1}{b}+\frac{1}{a},$$
$$\frac{2}{x}=\frac{a+b}{ab},$$
$$x=\frac{2ab}{a+b}.$$

4. There is no general formula for the sum of the terms of a harmonical progression.

EXERCISE XXI (E)

1. Extend each of the series given in § **5**, 1, by three terms.
2. Interpret the result of the example given in § **5**, 3.
3. Insert two harmonic means between 3 and 12.
4. Show that a, b and c are in

   (i) A.P. if $\frac{a-b}{b-c}=\frac{a}{a}$,

   (ii) G.P. if $\frac{a-b}{b-c}=\frac{a}{b}$,

   (iii) H.P. if $\frac{a-b}{b-c}=\frac{a}{c}$.
5. Construct and examine the graph of a harmonical progression.
6. In fig. 6, page 247, join AD, and through the point of intersection of AD and BC, draw a straight line parallel to a, and therefore to b, and terminated by AC and BD.

   Show that this straight line is the harmonic mean between a and b.

§ 6. Compound Interest.

In compound interest, at the end of a stated period, usually a year, the interest for that period is added to the principal, thereby giving a larger principal for the next period.

If interest is paid at the rate of, say, 5 % per annum, then a principal P, invested for one year, gains an interest of ·05P, and amounts therefore to 1·05P, i.e. 1·05 times the principal at the beginning of the year. If this is allowed to remain for another year, at the end of this second year the sum amounts to

$$1{\cdot}05\text{P} \times 1{\cdot}05, \quad \text{i.e. to } \text{P} \times (1{\cdot}05)^2,$$

and so on. Thus the amounts at the end of successive years are as follows:

Year	1	2	3	......	n
Amount	P(1·05)	P(1·05)^2	P(1·05)^3	......	P(1·05)^n

If r is the rate per cent. per annum, the amount at the end of the nth year is $\text{P}\left(1+\frac{r}{100}\right)^n$.

Notice that $\left(1+\frac{r}{100}\right)$ is the sum that £1 amounts to in one year. If we call this amount a, the amount of P in n years becomes $\text{P}a^n$, and the compound interest is therefore $\text{P}a^n - \text{P}$.

Example.—Find the sum to which £120 amounts in 5 years at 4 % per annum, compound interest.

$$\text{The amount (A)} = \text{P}\left(1+\frac{r}{100}\right)^5$$
$$= 120 \times (1{\cdot}04)^5.$$

This is best evaluated by logarithms. Thus:

$$\log \text{A} = \log 120 + 5 \log 1{\cdot}04.$$

The rest is easy.

Exercise XXI (f)

1. Find the compound interest on £250 for 3 years at 5 % per annum.
2. Construct graphs to contrast the simple and compound interest on, say, £100 for various periods at, say, 4 % per annum.
3. What sum will amount to £300 in 5 years at $2\frac{1}{2}$ % per annum?

4. The population of a town in 1880 was 150,000. It increased by 5 % each decade (10 years). What was the population in 1910?

5. The pressure in the bell-jar of an air-pump at the end of successive strokes was as follows:

Stroke - - - -	1st	2nd	3rd	4th	etc.
Pressure (lbs. per sq. in.)	15	13·5	12·15	10·935	etc.

Calculate the pressure at the end of the 20th stroke.

6. The resistance in a motor armature circuit when the starter is on the various studs is given by the series ·

$$R, \quad Rf, \quad Rf^2, \quad Rf^3 \ldots \text{etc.}$$

Calculate the resistance for 5 stops when R is 2 units and Rf^4 is 20 units.

7. The following are successive swings of a pendulum:

$$50, \quad 49, \quad 48{\cdot}02, \quad 47{\cdot}06, \quad \text{etc., cms.}$$

Calculate the length of the 10th swing.

8. In the "Achilles and Tortoise" race, if Achilles runs ten times as quickly as the tortoise, and the tortoise has 100 yards start, then when Achilles has covered 100 yards the tortoise has moved forward 10 yards, and so on.

Achilles has thus to cover a distance

$$(100 + 10 + 1 + \tfrac{1}{10} + \tfrac{1}{100} + \text{etc.})$$

before he catches up the tortoise, i.e. before the distance between him and the tortoise is 0. Find the distance by summing the series.

9. A farrier bargains to shoe a horse at a farthing for the first nail, a halfpenny for the second, a penny for the third, and so on.

If there are seven nails in each shoe, find the total cost for all four shoes, and the cost of the last nail. Compare the cost for the last shoe with that of the first.

§ 7. Application of Logarithms to Geometrical Progressions.

Let the series be a, ar, ar^2, $ar^2 \ldots ar^{n-1}$.

The logs of the terms are:

$$\log a, \quad (\log a + \log r), \quad (\log a + 2\log r), \ \ldots \ (\log a + \overline{n-1}\log r).$$

Notice that these terms have a common difference, namely, $\log r$. The terms are, therefore, in Arithmetical Progression.

The logarithms of the terms of a geometrical progression are in arithmetical progression, the common difference being the logarithm of the common ratio.

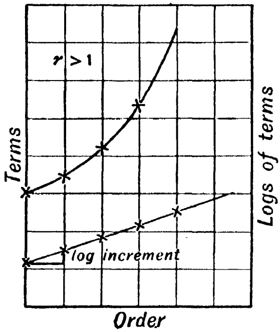


Fig. 10.

This result is of importance in Science.

If the common ratio is less than unity, the common difference of the A.P. is negative.

The relation between a G.P. and the corresponding A.P. of the logs of its terms is illustrated graphically in fig. 10.

The common difference between the logs of the terms is called the **Logarithmic Increment** if r is greater than unity, and the **Logarithmic Decrement** if less than unity.

Exercise XXI (G)

1. Plot the logarithms of the terms of the following series, and state the logarithmic increment or decrement in each case:

 (i) 1, 10, 100, 1000, etc. (ii) 100, 10, 1, 0·1, 0·01, 0·001, etc.

 (iii) 2, 10, 50, 250, etc. (iv) 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

2. The following are the angular displacements of a pendulum, from its position of rest, during successive swings:

Right	10		9		8·2		7·4	degrees.
Left		9·5		8·6		7·8		degrees.

 Find (*a*) the logarithmic decrement (i) per half swing, (ii) per full swing;

 (*b*) the angular displacement during its 10th excursion towards the left.

3. The heights to which a ball rises in successive rebounds are as follows:

 3 ft., 2 ft., 1 ft. 4 ins., $10\frac{2}{3}$ ins.

 What relation exists between these numbers?

4. Compare the second term of the series:

$$1, \ \tfrac{1}{2}, \ \tfrac{1}{4}, \ \tfrac{1}{8}, \text{ etc.},$$

with the sum of an infinite number of the terms following it.

5. Find the sum of varying numbers of terms of the series:

$$1, \ \tfrac{1}{3}, \ \tfrac{1}{9}, \ \tfrac{1}{27}, \ \tfrac{1}{81}, \text{ etc.},$$

and plot the results against the number of terms.

After examining the graph, say to what value the sum tends.

6. Find the value of the series:

$$1 - x + x^2 - x^3 + \text{etc.}, \quad \text{when } x < 1,$$

and show that it is the difference between the values of an infinite number of terms of each of the series:

$$1 + x^2 + x^4 + x^6 + \text{etc.} \quad \text{and} \quad x + x^3 + x^5 + \text{etc.}$$

7. The following numbers show the population of England and Wales for the years given.

Test whether they follow approximately the geometric law.

Year - - -	1851	1861	1871	1881	1891	1901	1911
Population (mill'ns)	17·93	20·07	22·71	25·97	29·00	32·53	36·07

8. There is an important series:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{(n-1)!} + \text{etc.},$$

where $3! = 1 \times 2 \times 3$, $4! = 1 \times 2 \times 3 \times 4$, etc.

Compare each of the terms after the first with the corresponding terms of the series:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{n-1}},$$

e.g. compare $\dfrac{1}{3!}$ with $\dfrac{1}{2^2}$.

Hence deduce that the sum of an infinite number of terms of the first series is less than 3.

9. Take the series $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \text{etc.}$, and find the ratio of the $(n+1)$th term to the nth term. How does this ratio change when n is increased indefinitely, x remaining constant? After what value of n will the terms decrease successively?

10 The following numbers are taken from a table, showing:
(1) The annuity £100 will purchase.
(2) The price of an annuity of £10.

Age - -	50	52	54	56	58	60	62	64
(1) £	6/6/9	6/11/6	6/17/1	7/3/4	7/10/7	7/19/0	8/8/11	9/0/6
(2) £	157/16	152/0	145/18	139/10	132/19	125/15	118/8	110/16

Plot each, (1) and (2), against age, and by interpolation find the figures for age 57, and by extrapolation the probable figures for ages 45 and 68.

Draw other conclusions, if possible.

11. Prove that the product of the sum of, and the difference between, consecutive terms of a G.P. form another G.P.

12. Find the average of n terms of a G.P., the first term of which is a and the common ratio r.

13. Find the average of n terms of an A.P., the first term of which is a and the common difference d.

14. Find the sum of x terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \text{etc.}$

By how much does the sum to infinity exceed the sum of x terms?

15. Determine the number of years in which a sum of money will double itself at 5 per cent. per annum compound interest.

16. The sum of the following series can be found by the same method as that for finding the sum of a geometric series. The expression obtained on subtracting contains a G.P.:

$$1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1}.$$

Find an expression for the sum of n terms.

Hence find the sum of ten terms of the following series:

$$1 + 6 + 27 + 108 + 405 + \text{etc.}$$

17. Write down the general term of the series:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.}$$

Find the ratio of the nth term to the preceding term. Under what conditions will the ratio be less than unity?

REVISION EXERCISE III

1. Plot the graph of $y=(x-2)(x+3)$ between $x=+3$ and $y=-4$, and use the graph to find approximately the roots of the equation
$$4x^2+4x-11=0.$$
State your construction, and give reasons for your inferences.

2. Plot the point $x=3$, $y=4$, and the straight line $y=2$. Draw the locus of a point which moves so that its distance from the point 3, 4, is equal to its perpendicular distance from the straight line $y=2$.

   Verify that the equation to the locus is $y=\frac{1}{4}x^2-\frac{3}{2}x+\frac{21}{4}$.

3. A peg top has the form of an equilateral cone, and a hemisphere placed base to base. If the diameter of the base of the cone is s, find the volume of the peg top.

   If the density of the peg top is 0·75, how far above the surface will the point be when the top is placed in water, point uppermost?

   When the density is 0·5, *the problem is more difficult, but is worth trying.*

4. Show that $\frac{\pi}{2}-\cos^{-1}\frac{x}{a}=\sin^{-1}\frac{x}{a}$.

5. Given that $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k,$$
   show that (i) $\dfrac{a+c+e}{b+d+f}=\dfrac{a}{b}$;

   (ii) $\dfrac{ab+cd+ef}{b^2+d^2+f^2}=\dfrac{a^2+c^2+e^2}{ab+cd+ef}=\dfrac{a}{b}$.

6. A passenger in a train travelling at 60 miles per hour observes that it takes 8 seconds to pass a train 110 yards long, going in the same direction. How long would it have taken if the trains had been travelling in opposite directions?

   If the first train were 132 yards long, in what time would they pass one another in each case?

7. A man puts by £100 at the beginning of each year to accumulate at Compound Interest at 4 per cent. per annum. Show that when he has put by his tenth instalment, he has accumulated a fund of £1200. [Assume $(1{\cdot}04)^{10}=1{\cdot}48$.]

8. The first two terms of an A.P. are $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{1+\sqrt{2}}$; find the third term.

   What would the third term be if the numbers were in G.P.?

9. If $\sin\theta=\dfrac{x}{a}$, show that $\cos\theta=\dfrac{\sqrt{a^2-x^2}}{a}$ and $\tan\theta=\dfrac{x}{\sqrt{a^2-x^2}}$.

10. Find how far a sphere of diameter 2″ will sink into a conical wine-glass, of which the depth and the diameter of the mouth are each $2\frac{1}{2}''$.

    Find also what volume of the sphere is inside the wine-glass.

CHAPTER XXII

GRAPHS OF VARIOUS FUNCTIONS, TRIGONOMETRY

§ 1. Graphs.

The considerations in previous chapters do not by any means exhaust the ways in which numbers may be related. We have already dealt with the cases in which one number varies as the first and as the second power, or both, and as the reciprocal of another.

With a knowledge of logarithms, we can now examine the case in which one number varies as any power of another.

Let $y = bx^n$, b and n being constants.

Then, taking logs of both sides,

$$\log y = n \log x + \log b.$$

Since the log of a constant is a constant, this is a linear equation; i.e. if $\log y$ and $\log x$ are plotted, the graph is a straight line. You may recognise this better if y' is written for $\log y$ and x' for $\log x$ and b' for $\log b$.

The equation then becomes

$$y' = nx' + b'.$$

The gradient of the graph is n, and its intersection with the axis of Y at b'.

Verify this by plotting the graph of the equation $y = 2x^3$:

-4	-3	-2	-1	$=x=$	0	1	2	3	4
-128	-54	-16	-2	$=2x^3=$	0	2	16	54	128

Observe that the graph:

(i) Is a curve with two prominent bends or elbows.

(ii) Passes through the origin.

Then take the logs of the positive numbers and plot them

$x=$	1	2	3	4
$\log x=$	0	0·301	0·477	0·602
$\log 2x^3=$	0·301	1·204	1·732	2·107

This graph is a straight line cutting the vertical axis at the point 0·301, and having a gradient 3.

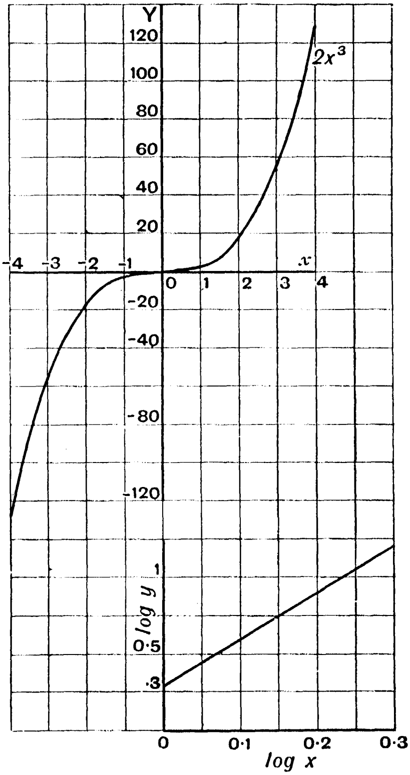


FIG. 1.

The equation is therefore

$$\log y = 3 \log x + 0{\cdot}301,$$

or

$$\log y = 3 \log x + \log 2,$$

which is the logarithmic equation of $y = 2x^3$.

Hence, to determine whether given numbers conform to a single-power law, simply plot the logs of the numbers, and see if a straight-line graph is obtained.

If there is an added constant in the expression, its effect will appear in the graph of the expression, and also in the graph of the log of the expression.

If $$y = bx^n + c,$$
then $$(y - c) = bx^n.$$

If n is positive, then, when x is 0, y is c. That is, the graph will cut the axis of Y at c.

This constant can then be subtracted from the numbers for y, and the logarithms plotted.

If n is negative, then, for bx^{-n} to be zero, x must be infinite, which means that the graph will approach but never meet the line $y = c$ (see page 110). This also will appear when the numbers are plotted.

If the logs of the expression $(bx^n + c)$ are plotted against $\log x$, you will find that the graph is not a straight line but a curve, which approaches the horizontal straight line drawn at a distance $\log c$ from the origin.

Summary.

The logarithmic method can be applied to any equation of the type $y = bx^n$.

When n is negative, the gradient of the log graph is downwards; when positive, upwards.

Application.

In an experiment upon gases, the following numbers for pressure and volume were obtained. Find the law connecting them.

Pressure (lbs. per sq. in.)	10	20	30	40	50	60
Volume (cubic ins.) -	16·03	9·802	7·355	5·996	5·119	4·498

Plot the logarithms of these numbers.

$\log p$	1	1·3010	1·4771	1·6021	1·6990	1·7782
$\log v$	1·205	0·9913	0·8666	0·7779	0·7092	0·6530

This graph is a straight line.

The gradient is $-\dfrac{\cdot 7782}{\cdot 552} = -1\cdot 4098 = -1\cdot 41$ (approx.).

The graph cuts the axis of $\log p$ at 2·7.

The equation is therefore $\log p = -1{\cdot}41 \log v + 2{\cdot}7$.
Now $2{\cdot}7$ is the log of $501{\cdot}2$

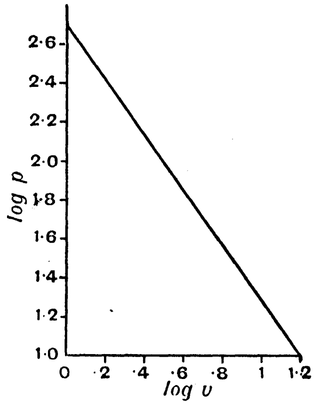


FIG. 2.

Hence the equation is $\quad p = 501{\cdot}2v^{-1{\cdot}41}$,

or $$p = \frac{501{\cdot}2}{v^{1{\cdot}41}},$$

or $$pv^{1{\cdot}41} = 501{\cdot}2.$$

Students of Engineering and Physics will recognise this as the adiabatic law.

EXERCISE XXII (A).

1. The following numbers show the fusing currents of fuses of the same material but of different diameters:

Diameter (inches) -	0·0072	0·0113	0·0149	0·0181	0·021
Fusing currents (amps.)	1	2	3	4	5

Find the law connecting the numbers. Remember that the log of a fraction is negative.

2. The table gives the pressure and volume of a pound of saturated steam. Find the relation.

Volume (cubic ft.) -	37·36	26·43	19·08	14·04	10·51	6·168
Pressure (lbs. sq. in.)	10·16	14·7	20·8	28·83	39·25	69·2

3. Find the equation connecting the following numbers:

$x = \frac{1}{2}$	1	2	3	4
$y = 16$	7	$4\frac{3}{4}$	$4\frac{1}{3}$	$4\frac{3}{16}$

4. Trace the graphs of the following functions of x:

(i) x^3. (ii) $(x-2)^3$. (iii) $(x-4)(x-2)(x+2)$.

(iv) $(x-4)(x-2)^2$. (v) $(x-4)(x^2+4)$.

How many turns or elbows has each graph?

5. (i) For what values of x does each expression in Exercise 4 equal 0?

(ii) How many roots has an equation containing x^3 as the highest power of x?

(iii) State the conditions under which some of the roots will be equal and under which some will be imaginary.

6. The expression (v), Ex. 4, is equal to -16 when $x=0$.

Determine for what other values of x the expression is equal to -16.

7. For what values of x has the expression $x^3 - 3x^2 - 4x + 8$ the value 8?

8. For what values of x has $3x^3 + 7x^2 - 6x + 12$ the value 12?

9. From the graphs of $2x-3$ and $\log_{10}x$, find the graph of $2x + \log_{10}x - 3$. Then solve the equation

$$2x + \log_{10}x - 3 = 0.$$

10. Solve graphically $y = x^3 - 4x^2 + 4x - 16$ and $y = 5x - 20$.

11. Plot the graphs of

$$\frac{x}{1+x}, \quad \frac{x}{1-x}, \quad \frac{1-x}{x}, \quad \frac{1+x}{x}, \quad \frac{1}{x-1}.$$

12. Draw a graph showing the sum of various numbers of terms of the series:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}, \text{ etc.}$$

Compute the value of the sum to three places of decimals.

13. Verify to two places of decimals that for the series:

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots,$$

the product of the values of the sum of the series when $x=1$ and when $x=2$, is equal to the value of the sum of the series when $x=1+2$, i.e. when $x=3$.

14. The following numbers are thought to follow the law $y=ab^{-x}$. If so, find the probable values of a and b. There are errors of observation.

x	0·1	0·2	0·4	0·6	1·0	1·5	2·0
y	350	316	120	63	12·86	2·57	0·425

15. If $y=20+\sqrt{30+x^2}$, take various values of x from 10 to 50, and calculate y.

Plot the results on squared paper, and find the straight line which most nearly agrees with these values of y. Write down the equation to this straight line.

16. By graphical means, find the roots between 1 and 2 of the following equation:

$$x^3+5x-11=0.$$

17. Plot the graphs of

(i) $y=0{\cdot}87\left(x+\dfrac{x^3}{300}\right)$, (ii) $y=10\log_{10}\left(\dfrac{10+x}{10-x}\right)$,

from $x=0$ to $x=7$.

18. The following numbers give the indicated horse-power, P, of a vessel at a speed of v knots:

v	20·1	24·9	30·2
P	1054	2135	3850

Test whether these numbers follow approximately the law $P=av^n$, and, if so, determine the approximate values of a and n.

§ 2. Graphs of the Trigonometrical Functions.

In Chapter IX, page 85, the trigonometrical ratios of angles between 0° and 90° have been considered. It is now necessary to consider angles of any magnitude.

Imagine a straight line OP (fig. 3) to rotate about the point 0 in, say, the counter-clockwise direction. Let OX be the initial direction of OP.

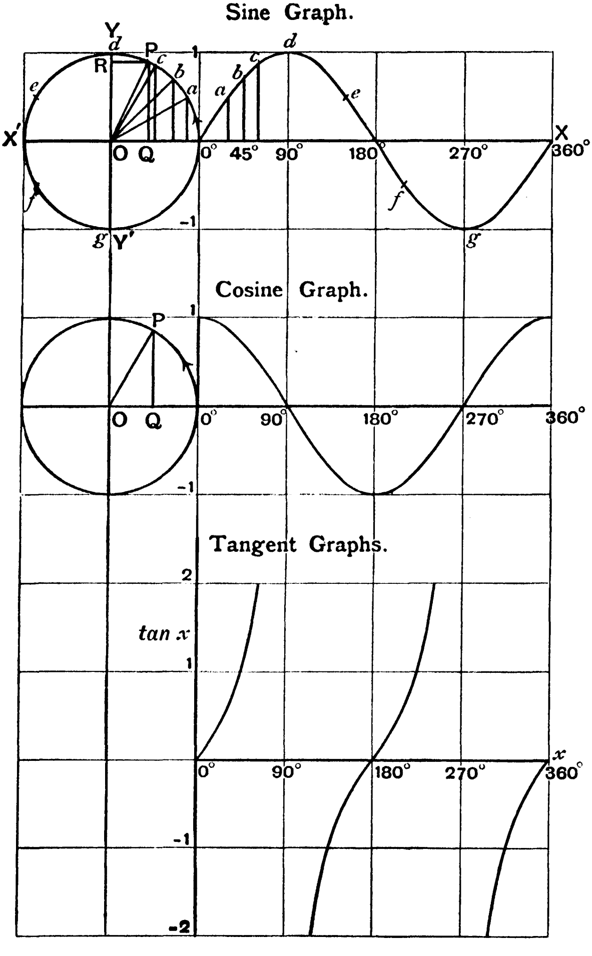


Fig. 3.

When the line has reached the position OP shown, it has rotated through the angle XOP.

Draw PQ at right angles to OX; then

$$\sin \angle XOP = \frac{PQ}{OP} \quad \text{and} \quad \cos \angle XOP = \frac{OQ}{OP}.$$

If PR is drawn at right angles to OY, then PQ = OR and

$$\sin \angle XOP = \frac{OR}{OP}.$$

Now, OR is the projection of the rotating line OP on the vertical line OY, and OQ the projection of OP on the horizontal line OX.

A more general definition of each ratio is as follows:

$$\text{sine} \angle XOP = \frac{\text{Projection of OP on the vertical axis}}{\text{Rotating line OP}},$$

$$\text{cosine} \angle XOP = \frac{\text{Projection of OP on the horizontal axis}}{\text{Rotating line OP}},$$

$$\text{tangent} \angle XOP = \frac{\text{Projection of OP on the vertical axis}}{\text{Projection of OP on the horizontal axis}}.$$

These definitions can be applied to angles greater than 90°.

The projections are positive if measured in the direction OX or OY, and negative if in the direction OX′ or OY′.

Note carefully, the following:

(1) When OP has the direction OX, the length of the projection on the vertical axis is 0, and that on the horizontal is equal to the length of OP.

Hence, $\sin 0° = 0, \quad \cos 0° = \frac{OP}{OP} = 1 \quad \text{and} \quad \tan 0° = 0.$

(2) When OP has rotated through 90°, and has therefore the direction OY, the length of the projection on the vertical axis is equal to the length of OP, and that on the horizontal axis is 0.

Hence, $\sin 90° = 1, \quad \cos 90° = 0 \quad \text{and} \quad \tan 90° = \frac{1}{0} = \infty.$

(3) When OP is in the second quadrant, i.e. when the angle XOP is obtuse, the projection of OP on the horizontal is negative. The cosine and tangent of angles between 90° and 180° are therefore negative.

(4) When OP is in the third quadrant, both projections are negative. The sine and cosine of angles between 180° and 270° are therefore negative.

(5) When OP is in the fourth quadrant, the projection on the vertical is negative.

The sine and tangent of angles between 270° and 360° are therefore negative.

The graphs of the sine, cosine and tangent of angles from 0° to 360° are shown in fig. 3. The angle is represented on the axis of x and the value of the function on the axis of y.

Verify the following equalities:

$$\sin(180° - x) = \sin x, \qquad \sin(180° + x) = -\sin x.$$
$$\cos(180° - x) = -\cos x, \qquad \cos(180° + x) = -\cos x.$$
$$\tan(180° - x) = -\tan x, \qquad \tan(180° + x) = \tan x.$$

Examine the graph of $\tan x$ in the neighbourhood of 90° and of 270°. Observe that for an angle slightly less than 90°, $\tan x$ is positive and numerically large, and for an angle slightly greater than 90°, $\tan x$ is negative and numerically large.

It will be seen that in passing through 90° and 270°, $\tan x$ changes sign from positive to negative, and that for these values of x, $\tan x$ is infinite.

§ **3**. The relation $\sin(A+B) = \sin A \cos B + \cos A \sin B$ is readily proved as follows, when A and B are together less than 180°.

Let A and B be angles of a triangle ABC.

Then

$$C = 180° - (A+B) \text{ and } \sin C = \sin\{180° - (A+B)\} = \sin(A+B).$$

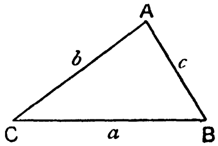


FIG. 4.

Now, $\quad c = a\cos B + b\cos A$ (see page 93), ...(i)

and since
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$c = \frac{a\sin C}{\sin A} \quad \text{and} \quad b = \frac{a\sin B}{\sin A}.$$

Hence, substituting these values in equation (i), we have:

$$\frac{a\sin C}{\sin A} = a\cos B + \frac{a\sin B}{\sin A}\cos A,$$

from which $\quad \sin C = \sin A\cos B + \sin B\cos A$

and $\quad \mathbf{\sin(A+B) = \sin A\cos B + \sin B\cos A.}$

From this, cos (A + B) may be found as follows:

$$\begin{aligned}\sin^2(A+B) &= \sin^2 A\cos^2 B + \sin^2 B\cos^2 A \\ &\qquad + 2\sin A\cos B\sin B\cos A \\ &= (1-\cos^2 A)\cos^2 B + \sin^2 B(1-\sin^2 A) \\ &\qquad + 2\sin A\cos B\sin B\cos A \\ &= \cos^2 B + \sin^2 B - \cos^2 A\cos^2 B - \sin^2 A\sin^2 B \\ &\qquad + 2\sin A\cos B\sin B\cos A \\ &= 1 - (\cos^2 A\cos^2 B + \sin^2 A\sin^2 B \\ &\qquad - 2\sin A\cos B\sin B\cos A),\end{aligned}$$

from which $1 - \sin^2(A+B) = (\cos A\cos B - \sin A\sin B)^2$

and $\cos^2(A+B) = (\cos A\cos B - \sin A\sin B)^2$

and $\mathbf{\cos(A+B) = \cos A\cos B - \sin A\sin B.}$

Again, $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Dividing above and below by $\cos A\cos B$ $\Bigg\} = \dfrac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}.$

$$\mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

§ 4. Functions of Negative Angles.

Fig. 5 shows angles $+A$ and $-A$ measured from the direction OO.

It is evident from the figure that the projection of OP on the vertical when in its final position after describing the angle $-A$ is opposite in sign to the corresponding projection when in the final position after describing the angle $+A$. Thus, in the given figure, ON is positive and OM negative.

On the other hand, the projections on the horizontal are exactly alike.

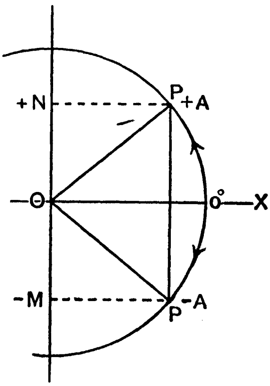


FIG. 5.

Hence,
$$\sin(-A) = -\sin A,$$
$$\cos(-A) = \cos A,$$
$$\tan(-A) = -\tan A.$$

If $-B$ is substituted for B in the equation,
$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$
the expansion of $\sin(A-B)$ is obtained.

Thus,
$$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$$
$$= \sin A \cos B - \cos A \sin B.$$

Similarly, $\sin(90° + A) = \sin\{90° - (-A)\} = \cos(-A) = \cos A.$

Exercise XXII (b)

1. Examine the sine graph and trace the change in the sine of an angle as the angle increases by, say, 15° from 0° to 360°.
2. Repeat Exercise 1 with the cosine and tangent graphs.
3. Examine the graphs of Exercises 1 and 2, and make a list of all the equalities you can find.
4. Write down the sin, cos and tan of the following angles:
   120°, 135°, 150°, 180°, 210°, 225°,
   240°, 270°, 300°, 315°, 330°, 360°.
5. By how many degrees is the cosine curve in advance of the sine curve?
6. Trace the graph of $2\sin x$, and on the same axes, the graph of $\cos x$. Then add the ordinates of the two graphs together and obtain another curve. It is the graph of $2\sin x + \cos x$.
7. Trace the graph of $2\sin x - \cos x$.
8. Construct the graph of $\sin^2 x$, i.e. the square of the sine of x, and also the graph of $\sin(x)^2$, i.e. the sine of the square of x.
9. Draw the graph of $\log_{10}\sin x$.
10. Procure a thin paste-board tube, and cut it across its axis at an angle, slit one part along its length, open it out flat, place it flat on squared paper, and draw a pencil line on the paper along the curved edge of the open tube. What kind of a curve does it appear to be? Verify by measurements.

11. Draw the graphs of (i) $\operatorname{cosec} x$, (ii) $\sec x$, (iii) $\cot x$, from $x = 0°$ to $x = 360°$.

12. Turning to fig. 3, the diagram of the rotating radius is usually called the clock diagram of the graph.

In Questions 6 and 7, show the clock diagram of each graph, including the resultant graph.

13. On the same axes, plot the curves $\sin x$ and $\sin (x + 30°)$.

On the same clock diagram, draw pointers to represent the rotating radii.

Determine the resultant curve, and insert in the clock diagram the pointer corresponding to this curve. Join the free end of this pointer to the free ends of the other pointers, and see what figure is obtained (fig. 6).

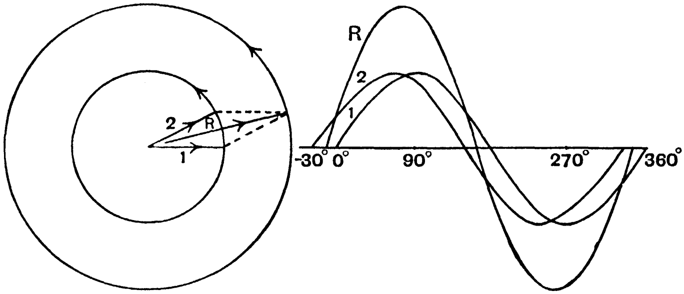


FIG. 6.

14. Plot the graphs of $\sin 2x$ and $\sin x$ on the same axes.

Draw also the clock diagram as in the previous question.

Determine the resultant graph, and insert in the clock diagram the corresponding pointer.

15. Turn to the relations given in § 3, and substituting $-B$ for B, show that:

$$\text{(i) } \sin (A - B) = \sin A \cos B - \cos A \sin B.$$

$$\text{(ii) } \cos (A - B) = \cos A \cos B + \sin A \sin B.$$

$$\text{(iii) } \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

16. Simplify the following:

(i) $\sin (A + B) + \sin (A - B)$. (ii) $\sin (A + B) - \sin (A - B)$.

(iii) $\cos (A + B) + \cos (A - B)$. (iv) $\cos (A + B) - \cos (A - B)$.

(v) $\tan (A + B) + \tan (A - B)$. (vi) $\tan (A + B) - \tan (A - B)$.

CHAPTER XXIII

AN INTRODUCTION TO THE DIFFERENTIAL AND INTEGRAL CALCULUS

§ 1. Rate of Change of Simple Functions.

Consider first a linear function, i.e. a function the graph of which is a straight line.

Let y be a linear function of x, the precise relation being $y = ax + b$, a and b being constants.

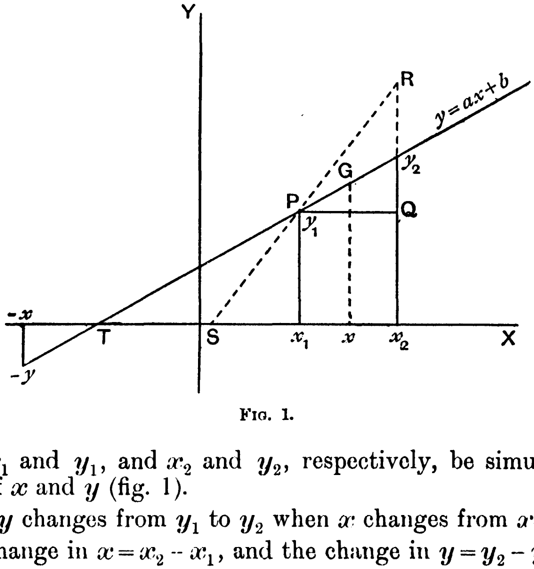


FIG. 1.

Let x_1 and y_1, and x_2 and y_2, respectively, be simultaneous values of x and y (fig. 1).

Then y changes from y_1 to y_2 when x changes from x_1 to x_2.

The change in $x = x_2 - x_1$, and the change in $y = y_2 - y_1$.

The ratio of the change in y to the change in x is, $\dfrac{y_2 - y_1}{x_2 - x_1}$.

It is already known that, in the case of the straight-line graph, this ratio is constant for all values of x_1 and x_2. Moreover, since $\triangle\, y_2QP$ is similar to $\triangle\, Px_1T$,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x_1P}{x_1T},$$

which is the tangent of the angle x_1TP, i.e. of the angle the

graph makes with the axis of X, and which we have seen is a measure of the gradient of the graph.

In this case, the tangent is positive, and this is true even when the graph is produced below the axis of X, for then both x and y are negative, and the quotient therefore positive.

We may say that the quotient $\frac{y_2 - y_1}{x_2 - x_1}$ represents the gradient at the point G, and has been determined by taking two values of x, one on each side of that for G, and finding the quotient,

$$\frac{\text{Change in } y}{\text{Change in } x}.$$

When these changes, and therefore the sides of the $\triangle PQy_2$, are small, it is easier to find the quotient from $\triangle Px_1T$, or better, from $\triangle GxT$, which extends to the very point G.

This quotient represents the **rate** at which y, i.e. $ax+b$, changes as x is changed, and, in order to understand the full significance of this, compare two linear graphs passing through P.

In fig. 1, *it is seen that, for the graph* PR, *the change in* y, *viz.* QR, *when the change in* x *is* $(x_2 - x_1)$, *is greater than that for the graph* TP, *viz.* Qy_2, *for the same change in* x. *The gradient of* PR *is steeper than that of* TP, *the difference being*

$$\frac{QR - Qy_2}{x_2 - x_1}.$$

It is seen from the figure that this difference is equal to

$$\tan \angle QPR - \tan \angle QPy_2,$$

and since $\quad \angle QPR = \angle XSP, \quad$ *and* $\quad \angle QPy_2 = \angle XTP,$

to $$\tan \angle XSP - \tan \angle XTP.$$

By Geometry, extr. $\angle XSP$ *is greater than* opp. intr. $\angle XTP$.

Returning to the relation $y = ax + b$, we know from the chapter on the linear graph that the gradient is a.

The rate of change of y, i.e. $ax+b$, as x changes, is therefore a.

The points to remember are:

(i) In the straight-line graph the gradient is constant, and is measured by the tangent of the angle the graph makes with the axis of x.

(ii) The gradient of the graph measures the rate at which y changes with x, and that in the straight-line graph this rate is constant.

§ **2.** When the changes are very small, it is usual to employ special symbols to represent them. Thus a small change in x is represented by δx, and the corresponding change in y by δy.

The Greek letter δ (d) is not in this case a multiplier, but when placed before another letter, it must be taken to indicate a small change in the quantity represented by the letter. Referring to the figure, in such a case PQ *represents* δx, *and* Qy_2 δy, *and the triangle* PQy_2 *may be as small as we please.*

The quotient $\dfrac{\text{change in } y}{\text{small change in } x}$ is then written $\dfrac{\delta y}{\delta x}$.

If $y = ax + b$, we now know that $\dfrac{\delta y}{\delta x} = a$, the gradient of the straight-line graph.

EXERCISE XXIII (A)

1. Draw a graph representing the distance covered by a moving body, as shown in the following table:

Time (secs.) -	0	1	2	3	4	5	6
Distance (feet)	0	3	6	9	12	15	18

Find the rate of change of distance with time, at definite instants (say at $1\frac{1}{2}$, 2, $3\frac{1}{4}$, etc., seconds after time 0), by taking small intervals containing the instant and determining the quotient $\dfrac{\text{change in distance}}{\text{change in time}}$.

The quotient, as you know, is the velocity of the body. What do you know of the velocity in this case?

2. Draw the graphs of

(i) $2x + 5$, (ii) $2x$, (iii) $5 - 2x$,

(iv) $-2x$, (v) $\frac{1}{4}x - 2$, (vi) $\frac{1}{4}x + 2$,

and in each case determine at various points the rate at which the value of the expression changes with respect to x.

3. What is the gradient of the graph $y = 0x + b$?

What therefore is the rate of the change of y with respect to x?

4. Take the function $y = 2x + 3$, and find the value of y when x is, say, 3. Then increase the value of x to, say, 3·1, and find the new value of y.

   Now calculate the quotient $\frac{\text{change in } y}{\text{change in } x}$.

   Repeat the process with x equal to, say, -4 and $-4{\cdot}05$, or $-3{\cdot}95$.

   Compare the quotients.

   Repeat this exercise with other linear functions of x.

§ 3. Rate of Change of Non-Linear Functions.

When we consider functions which are not linear, the determination of the rate of change of the function is not so easy.

The more important functions, other than the linear function of which we have drawn graphs, are:

(i) $y = ax^2$, (ii) $y = ax^2 + c$, (iii) $y = ax^2 + bx + c$,

(iv) $y = \pm\sqrt{a^2 + x^2}$, (v) $y = \frac{1}{x} + c$,

and (vi) the trigonometrical functions.

The graphs of these are curves.

Let the curve in fig. 2 represent a portion of the graph of any one of these functions, and let us say that we wish to determine the rate of change of the function in the neighbourhood of the value denoted by the point G. Let y_1 and y_2 be the values of the function, when the values of x are x_1 and x_2, such that the value of x corresponding to G is between x_1 and x_2. Then if x_1 and x_2 do not differ much from x, the quotient $\frac{y_2 - y_1}{x_2 - x_1}$, if not exact, is an approximation to the rate at which the function changes in the neighbourhood of G. As in the case of the linear function, this quotient is a measure of the tangent of the angle which the straight line joining y_2, and y_1, makes with the axis of x (i.e. tan $\angle x_1 T y_1$).

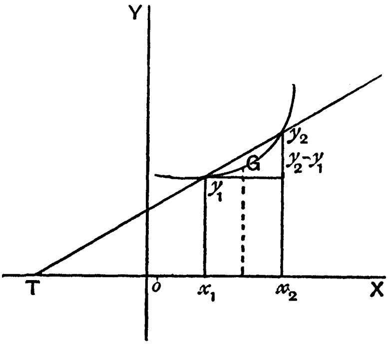


FIG. 2.

Exercise XXIII (b)

1. Plot the graph $y=x^2$, and in the manner indicated in § **3**, determine the approximate rate of change of y for different portions of the graph.

   Tabulate as follows:

Neighbourhood of $x=$	$x_1=$	$x_2=$	$(x_2-x_1)=$	$y_1=$	$y_2=$	$y_2-y_1=$	$\frac{y_2-y_1}{x_2-x_1}=$
+1	0·8	1·2	0·4	0·6	1·4	0·8	$\frac{0·8}{0·4}=2·0$
+2							
+3							
−1							
−2							
−3							

   Plot these quotients against the values given in the first column, and draw your conclusions.

2. Plot the graph of $y=\frac{1}{x}$, from, say, $x=1$ to $x=3$, and in determining the quotient $\frac{y_2-y_1}{x_2-x_1}$, in the neighbourhood of $x=2$, find the effect of making x_1 and x_2 more and more nearly equal to 2. For example, in the first case take $x_1=1$ and $x_2=3$, in the next $x_1=1·2$ and $x_2=2·8$, and so on until the measurements are too small to be reliable. Enter your results in a table like the following:

   Neighbourhood of $x=2$.

x_1	x_2	x_2-x_1	y_1	y_2	y_2-y_1	$\frac{y_2-y_1}{x_2-x_1}$
1	3	2	1	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}=-0·333$
1·2	2·8	1·6	0·83	0·35	−0·48	−0·3
1·4	2·6	1·2				
1·6	2·4	0·8				
1·8	2·2	0·4				
1·9	2·1	0·2				

   Examine the numbers carefully, and draw your conclusions. You will find it helpful to plot the numbers against (x_2-x_1), and by producing the graph to determine $\frac{y_2-y_1}{x_2-x_1}$ when (x_2-x_1) is 0.

It is worth noting that the gradient of the graph for the negative values of x has the same sign as that of the graph for positive values.

3. Repeat Exercise 2 with the following graphs:

(i) $y = x^2$. (ii) $y = \pm\sqrt{25 - x^2}$. (iii) $y = 9 - 2x^2$.

4. Determine $\frac{y_2 - y_1}{x_2 - x_1}$ for the functions given in Exercises 2 and 3, by calculating y_2 and y_1 from the equation connecting y and x. Plot the value of the quotient as suggested in Exercise 2, and find the value to which it tends as $x_2 - x_1$ approaches 0.

§ **4.** In Exercise 2, you probably found that as the values of x_1 and x_2 approached the value of x, measurement became more and more difficult. You could, however, have determined the value of the quotient from the triangle formed by producing y_2y_1, to cut the axis of x at T, or from the tangent of the angle x_1Ty_1 (fig. 2).

You doubtless noticed that as x_1 and x_2 became more nearly equal to one another, the points y_1 and y_2 approached the point G. When the points are very close to G, it is better to refer to $\triangle x_1Ty_1$ for the quotient. The question is, can we draw a triangle corresponding to x_1Ty_1 when y_1 and y_2 actually reach G? Well, the straight line y_2y_1T is gradually approaching a position such that, instead of cutting the curve, it only touches it, and when finally it has reached this position it will be a **tangent** to the curve at point G.

If $\frac{\delta y}{\delta x}$ represents $\frac{\text{change in } y}{\text{change in } x}$ before y_1 and y_2 reach G, but are very close to it, we know that $\frac{\delta y}{\delta x}$ can be found from the angle x_1Ty_1. If the same relation holds, then, when y_1 and y_2 actually reach G, the value of $\frac{\delta y}{\delta x}$, **although the terms δy and δx are smaller than any measurable quantity,** can be found from the angle the tangent GT makes with the axis of x. In fact, we have seen that it is the trigonometrical tangent of the angle, the tangent GT makes with the axis of x. (See footnote to § **2,** Chapter X.)

The value of the quotient $\frac{\delta y}{\delta x}$, when the terms reach this limit, is written $\frac{dy}{dx}$, which must be regarded merely as a **symbol**

denoting a limiting value of a quotient, or the value to which a quotient tends when the terms are diminished indefinitely, and not as consisting of separate algebraic numbers.

The tangent at the point G actually shows the direction of the curve, or its gradient at this point. Hence, we can determine the rate of change of a function when it has various values, from its graph, by drawing tangents to the graph at different points corresponding to the values in mind.

It is now an easy matter to trace from its graph the rate of change of a given function.

EXAMPLE. $y = ax^2 + c$.

To show the rate of change of y with x, draw tangents to the graph at various points. Thus, in fig. 3, starting at V and examining the curve to the right, draw tangents at V, A, B, C,

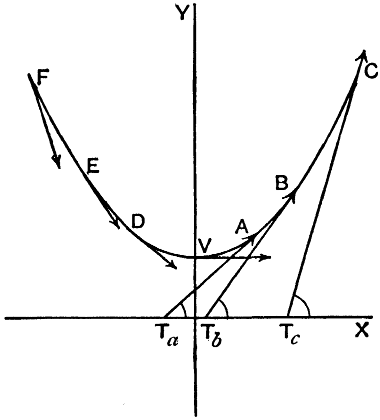


FIG. 3.

etc. At V, the tangent is parallel to the axis of x; the gradient is therefore 0. The gradient at A is given by $\tan \angle T_a$, at B by $\tan \angle T_b$, at C by $\tan \angle T_c$. The rate of change is positive, and increases in value as x increases. The tangent, however, never reaches a direction perpendicular to the axis of x.

To the left of V we have a similar result, but the gradient is in this case negative. Or, starting from F, the gradient is negative, but approaches 0, which it reaches at V, and afterwards is positive.

Exercise XXIII (c)

1. Plot the graph of $y = 3x^2 + 2$, and by the method of tangents, determine the gradient at points corresponding to various values of x, positive and negative. Plot the values obtained and the values of x, and draw your conclusions.

2. The distance s, through which a stone falls, is given by the formula $s = \frac{1}{2}gt^2$, where t = the time in seconds and $g = 32$. Plot the graph, and from it find the rate of change of distance with time, $\left(\frac{ds}{dt}\right)$, i.e. the velocity, at the following instants reckoned from the time of release, 0, 1 second, 2 seconds, 3 seconds, 4 seconds, 5 seconds.

   Then plot velocity and time, and draw your conclusions. What do you know of the velocity in this case?

§ 5. Differentiation and Integration.

In Exercise 1 you have found that when the rate of change of the function $y = 3x^2 + 2$ is plotted, the graph is an inclined straight line.

That is, $\frac{dy}{dx}$ also, is a function of x, and is of the form $ax + b$.

You will find it a good plan to arrange these graphs so that the graph of the rate of change is directly below that of the function (see fig. 4).

In the case given, the equation to the straight-line graph is,

$$\textit{rate of change} = 6x,$$

$$\text{i.e.}\quad \frac{dy}{dx} = 6x.$$

We have previously (page 197) referred to a very important relation between these graphs.

Draw ordinates at points $x = 1$ and $x = 4$ in each graph.

Find the area between these ordinates in the $\frac{dy}{dx}$ graph, and compare it with the difference between the ordinates in the graph of y.

Repeat the operation at values $x = 0$ and $x = 1$, $x = 4$ and $x = -2$, etc.

You will notice that the $\frac{dy}{dx}$ graph enables you to find the parts of the ordinates of the y graph above the level of the

vertex of the parabola. It does not give you the constant 2, but only the part $3x^2$ of the expression $3x^2+2$. One full ordinate must be known before the constant can be found.

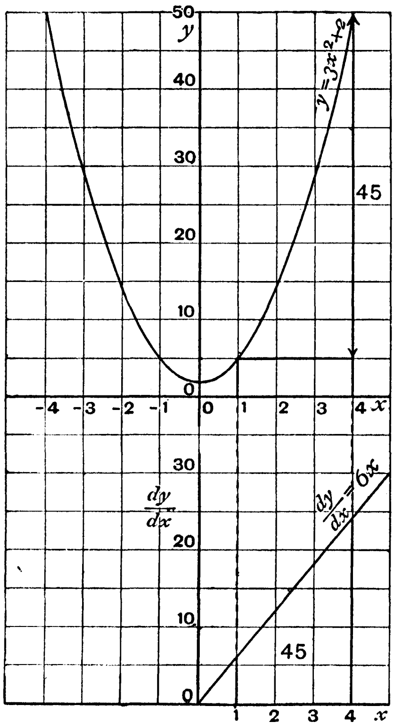


FIG. 4.

These processes are very important. That of obtaining the rate of change of a function is called **Differentiation**, and that of obtaining the function, or that part of it which is not constant, from its rate of change, **Integration**.

The sign of the former, viz. differentiation, is $\frac{d}{dx}$, and that of integration, $\int \ldots dx$.

The dx in each case indicates that the differentiation or integration is made with respect to x.

The value of $\frac{dy}{dx}$ for any function of x, represented by y, is usually called the differential coefficient of the function, or simply the differential.

When the value of an **integral** between two **definite** values of the variable is required, the values of the variable, say a and b, are shown thus:

$$\int_a^b \dots dx.$$

A reference to the figure will show that the integral from a to b is the difference between the integrals from 0 to b and from 0 to a.

Thus, in the $\frac{dy}{dx}$ graph of fig. 4, the area between the ordinates $x = 1$ and $x = 4$ is:

$$\int_1^4 6x\,dx = \int_0^4 6x\,dx - \int_0^1 6x\,dx$$

$$= \begin{pmatrix} 3x^2 + 2 \\ \text{when } x = 4 \end{pmatrix} - \begin{pmatrix} 3x^2 + 2 \\ \text{when } x = 1 \end{pmatrix}.$$

As the constant disappears on subtracting, it is usual to write the result in the following form:

$$\int_1^4 6x\,dx = {}_1^4\Big[3x^2\Big]$$

$$= [3 \times (4)^2 - 3(1)^2]$$

$$= 45.$$

Exercise XXIII (d)

1. Determine graphically the differential of $y = x^2 - 3x + 2$.
   Draw a new axis of y, call it y', passing through the vertex of the parabola, and then refer the result to the original axes in the manner given on page 165 (see fig. 6).

2. Determine $\frac{dy}{dx}$ when y represents the following functions of x:
   (i) $-x^2$. (ii) $-3x^2$. (iii) $-2x^2 - 3$.
   (iv) $-2x^2 + 3$. (v) $2x^2 + 5x - 3$. (vi) $3 - 5x - 2x^2$.
   Then state the rule for finding the differential of a function of the type $ax^2 + bx + c$.

3. **Examine the rate of change of the sine and of the cosine graphs. Use the tangent method, graph the results, and draw your conclusion.**

4. Draw the graph of $\frac{1}{x}$, and determine the gradient at various points. Plot the results, and by means of logarithms, find the law of the differential.

§ 6. In a few cases the differential coefficient can be obtained from simple considerations.

EXAMPLE i.—The graph (fig. 5) is that of the function

$$y = ax^2 + c. \quad \text{..............................(i)}$$

To find the gradient at a point G.

Let x and y be the co-ordinates of G.

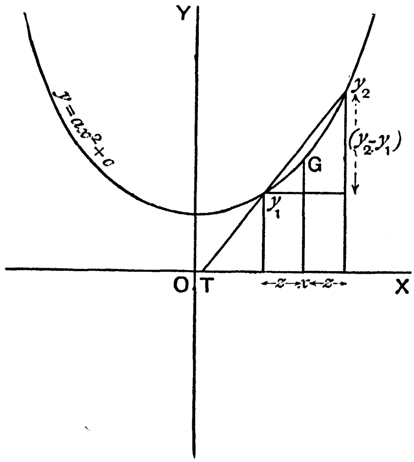


FIG. 5.

Take two other values of x, one less by z and the other greater by z than x, and let y_1 and y_2 be, respectively, the values of y.

Then $\quad y_1 = a(x - z)^2 + c$
and $\quad y_2 = a(x + z)^2 + c$ $\Big\}$ from (i).

Now, if xG is produced, it will cut the straight line joining y_2, y_1 at its middle point, and this is true for all chords y_2y_1, however short, obtained by taking pairs of ordinates, the feet of which are equidistant from x.

Moreover, the difference in the ordinates of any such pair of points is:
$$y_2 - y_1 = a(x+z)^2 + c - \{a(x-z)^2 + c\}$$
$$= 4axz;$$

$\therefore$ the gradient of all such chords is $\frac{4axz}{2z} = 2ax$.

It follows that all such chords make the same angle with the axis of x, and are therefore parallel.

The same relation holds when y_2 and y_1 reach the point G, and y_2y_1T becomes the tangent at G.

Hence the gradient at any point G is $2ax$.

It follows that the rate of change of the function $ax^2 + c$ with respect to x is $2ax$, and is, of course, itself a function of x. For example, when $x = 3$, the value of the function is $9a + c$, and its rate of change at this value of x is $2 \times a \times 3 = 6a$.

Hence, if $y = ax^2 + c$, $\frac{dy}{dx} = 2ax$.

It follows that the integral of $2ax$ is $ax^2 + c$,

$$\text{i.e.} \quad \int 2ax\,dx = ax^2 + c \quad \text{or} \quad \int ax\,dx = \tfrac{1}{2}ax^2 + c.$$

EXAMPLE ii.—The differentiation of the general expression,
$$y = ax^2 + bx + c,$$
can be accomplished by the method of Example i.

Move the axis of y (fig. 6) so that it passes through the vertex, i.e. to the point
$$x = -\frac{b}{2a} \text{ (see Chap. XVII, § 3);}$$
then the values of the new x, viz. x_1, are given by
$$x_1 = x - \left(-\frac{b}{2a}\right), \quad \text{from which} \quad x = x_1 - \frac{b}{2a}.$$

The expression in terms of x_1 becomes
$$y_1 = a\left(x_1 - \frac{b}{2a}\right)^2 + b\left(x_1 - \frac{b}{2a}\right) + c$$
$$= ax_1^{\,2} - \frac{b^2 - 4ac}{4a}.$$

The gradient of this we have seen to be $2ax_1$.

Referred to the original axes, we have

$$2ax_1 = 2a\left\{x - \left(-\frac{b}{2a}\right)\right\} = 2ax + b,$$

i.e. $$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b.$$

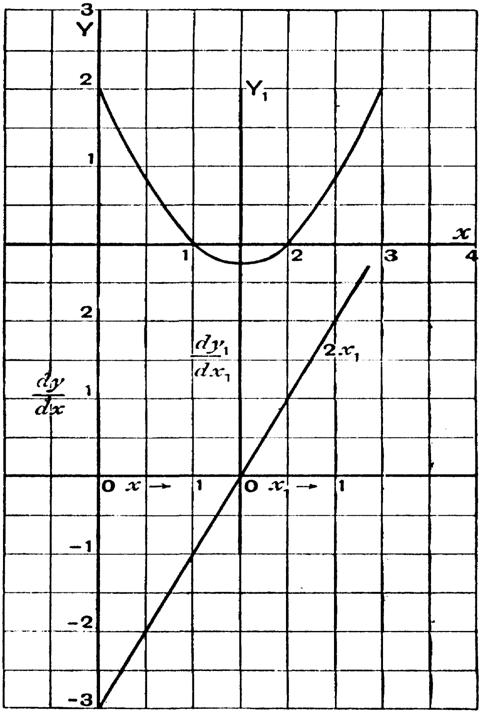


Fig. 6.

It follows also that

$$\int (2ax + b)\,dx = ax^2 + bx + c.$$

Exercise XXIII (e)

1. Similarly show that

$$\frac{d}{dx}(ax^2 - bx + c) = 2ax - b,$$

that $$\frac{d}{dx}(-ax^2+bx+c) = -2ax+b,$$

and that $$\frac{d}{dx}(c-bx-ax^2) = -b-2ax.$$

2. If $y = x^2 - 3x + 2$, find $\frac{dy}{dx}$.

Then find the value of $\frac{dy}{dx}$ when $x = 2$.

The result represents the value of the gradient at the point on the curve for which $x = 2$, and is therefore the value of the tangent of the angle which the geometrical tangent at this point makes with the axis of x. Determine the angle in this case.

3. If $y = 12 + 8x - 3x^2$, find $\frac{dy}{dx}$, and determine the value of x for which $\frac{dy}{dx} = 0$.

Find also the gradient of the graph of y at the point at which $x = 1$, and at the point at which $x = 2$. Contrast the gradients at these two points.

4 Show that
$$\frac{d}{dx}(ax^2+bx+c) = \frac{d(ax^2)}{dx} + \frac{d(bx)}{dx} + \frac{d(c)}{dx},$$

and that $$\int(2ax+b)\,dx = \int 2ax\,dx + \int b\,dx.$$

5. Find the following integrals:
$$\int 6x\,dx, \quad \int 6\,dx, \quad \int(3x-2)\,dx, \quad \int(a-t)\,dt.$$

6. If $v = u + at$, find $\int v\,dt$.

EXAMPLE iii. $$y = \frac{1}{x} + c.$$

To find the gradient of the graph (fig. 7), take two values of x, say x_1 and x_2, and let the corresponding values of y be y_1 and y_2 respectively.

Then $$y_1 = \frac{1}{x_1} + c \quad \text{and} \quad y_2 = \frac{1}{x_2} + c.$$

If x_1 and x_2 are close together, the gradient of this part of the graph is approximately,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(\frac{1}{x_2} + c\right) - \left(\frac{1}{x_1} + c\right)}{x_2 - x_1}$$

$$= \frac{\frac{1}{x_2} - \frac{1}{x_1}}{x_2 - x_1}$$

$$= \frac{\frac{x_1 - x_2}{x_1 x_2}}{x_2 - x_1}$$

$$= -\frac{1}{x_1 x_2}.$$

This is, again, the tangent of the angle, $y_1 y_2 T$ makes with the axis of x.

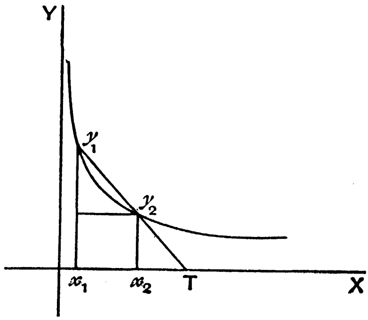


FIG. 7.

Now consider x_1 and x_2 to approach one another (and consequently y_1 and y_2 also), and finally to coincide at a value x.

Then $y_1 y_2 T$ becomes the tangent to the graph at a point, the coordinates of which may be called x and y.

The value of the trigonometrical tangent of the angle the tangent to the graph makes with the axis of x becomes

$$-\frac{1}{xx}, \quad \text{i.e.} \quad -\frac{1}{x^2}.$$

Hence, if $y = \frac{1}{x} + c$, $\qquad \frac{dy}{dx} = -\frac{1}{x^2}$,

and consequently,

$$\int -\frac{1}{x^2}\,dx = \frac{1}{x} + c \quad \text{or} \quad \int \frac{1}{x^2}\,dx = -\frac{1}{x} + c.$$

EXERCISE XXIII (F)

1. Plot the graph of $-\frac{1}{x^2}$, and contrast it with that of $\frac{1}{x}$.

2. In a manner similar to that in which $y=\frac{1}{x}+c$ was examined, show that if $y=\frac{a}{x^2}+c$, $\frac{dy}{dx}=-\frac{2a}{x^3}$, and that therefore

$$\int \frac{a}{x^3}\,dx = -\frac{a}{2x^2}+c.$$

3. Find the gradient of the curve $y=\frac{3}{x}+2$ at the point at which $x=2$.

EXAMPLE iv. $y=x^n$.

Take two values of x, say x_1 and x_2 as before, and let the corresponding values of y be y_1 and y_2; then

$$y_1 = x_1^n \quad \text{and} \quad y_2 = x_2^n.$$

The gradient is approximately,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2^n - x_1^n}{x_2 - x_1}$$

$$= \underbrace{x_2^{n-1} + x_2^{n-2}x_1 + x_2^{n-3}x_1^2 + \ldots + x_2x_1^{n-2} + x_1^{n-1}}_{n \text{ terms}}$$

(see Ch. XII, § 4).

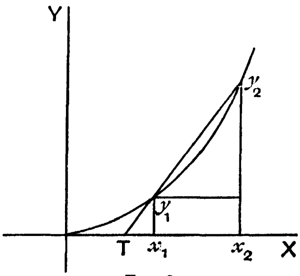


FIG. 8.

This is the value of the trigonometrical tangent of the angle the line y_2y_1T makes with the axis of x (fig. 8).

Now let x_1 and x_2 approach each other and finally reach coincidence at x. Then each term of the expression for the gradient becomes x^{n-1}

$$(\text{e.g. } x_2^{n-4}x_1^3 = x^{n-4}x^3 = x^{n-1}),$$

and the sum of n such terms is, of course, nx^{n-1}.

Hence, if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$.

It follows also that

$$\int nx^{n-1}\,dx = x^n + c \quad \text{or} \quad \int x^{n-1}\,dx = \frac{1}{n}x^n + c.$$

Exercise XXIII (g)

1. Apply these formulae to the cases $y = \frac{1}{x}$ or x^{-1}, $y = \frac{1}{x^2}$. What is your conclusion?

2. If $y = ax^n + c$, show by the method used above that

$$\frac{dy}{dx} = anx^{n-1}.$$

Write down the corresponding integral.

3. If $y = 3x^3$, find $\frac{dy}{dx}$ and $\int y\,dx$.

4. If $y = 3x^4 - 5x^3 + 2x^2 - 1$, find $\frac{dy}{dx}$ by differentiating each term.

5. If $y = x^2 - 3x + 2$, find $\int y\,dx$.

6. If $y = 6 + 3x - 2x^2$, find $\int y\,dx$. 7. If $y = ax^n$, find $\int y\,dx$.

§ 7. Sine and Cosine Graphs.

Up to the present we have generally measured angles in terms of the unit called a degree.

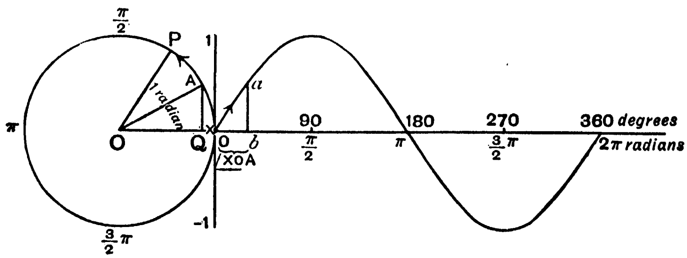


Fig. 9.

There is, however, another unit, the *radian*. The angle at the centre of a circle is proportional to the arc between the ends of

the radii containing the angle. If the length of the arc (measured along the curve) is equal to the radius, the angle is one *radian* (see $\angle$ XOP in fig. 9).

Since the radius is contained 2π times in the circumference, there are 2π *radians* in 360° or a cycle, or π *radians* in 180° or a straight angle. You know that $\pi = 3{\cdot}1416$ approx.

It is often more convenient to measure angles in radians.

The following exercise is interesting.

Consider the angle XOA; its sine is $\frac{\text{AQ}}{\text{OA}}$.

The angle itself is measured by $\frac{\text{arc AX}}{\text{radius OX or OA}}$ (radians).

The arc AX is longer than the side AQ, but as $\angle$XOA gets smaller, the difference changes.

Work the following exercise:

Measure the ordinate ab representing the sine of the angle represented by ob (in radians), and find the quotient $\frac{ab}{ob}$. Continue this operation, decreasing the angle ob by convenient amounts.

Examine the numbers. To what value do they tend?

Angle ob	Ordinate ab	Quotient $\frac{ab}{ob}$
$\frac{\pi}{10} = {\cdot}314$ radian.	·3	$\frac{\cdot 3}{\cdot 31}$

As the angle approaches 0, the quotient $\frac{ab}{ob}$ tends to the value 1. That is, the gradient of the sine curve in the neighbourhood of 0 is tending to the value 1. In other words, the tangent of the angle, which the tangent to the sine curve at the point $x=0$ makes with the axis of x, is 1. The angle is therefore 45°. The gradient of the sine curve varies from 1 to 0 as the angle increases from 0 to $\frac{\pi}{2}$, and previous practical work has suggested that its values when plotted give a cosine curve,

$$\text{i.e. } \frac{d(\sin x)}{dx} = \cos x.$$

Notice that the cosine curve is an angle $\frac{\pi}{2}$ (radians) in advance of the sine curve in its values (fig. 10).

Similarly, on plotting the gradients of the cosine curve, the sine curve is obtained, but with its negative portion first, because the gradient of the cosine curve from 0 to π is negative (fig. 10),

$$\text{i.e.}\quad \frac{d\cos x}{dx} = -\sin x.$$

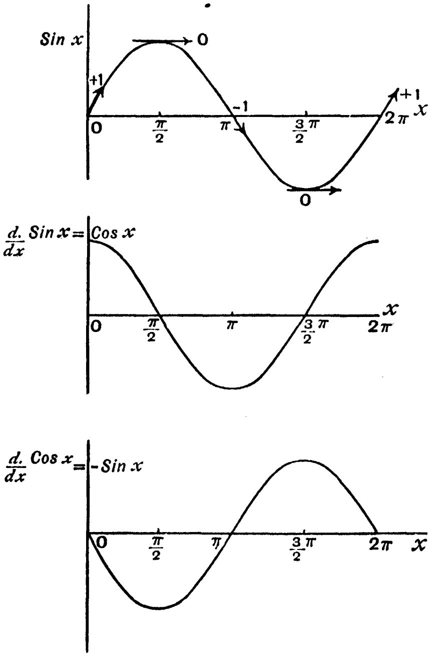


FIG. 10.

Notice, again, that the graph of the differential has its values an angle $\frac{\pi}{2}$ in advance of the graph of the function, viz. $\cos x$.

It follows that

$$\int \cos x\, dx = \sin x + c, \quad \text{and} \quad \int -\sin x\, dx = \cos x + c,$$

or

$$\int \sin x\, dx = -\cos x + c.$$

EXERCISE XXIII (H)

1. Find the value of $\frac{d \,.\, \sin x}{dx}$, when $x = 60°$.

2. Find the value of $\frac{d \,.\, \cos x}{dx}$, when $x = \frac{3\pi}{2}$ (radians).

3. Determine the following integrals:

   (a) $\int_0^{\pi} \sin x\, dx$. (b) $\int_0^{2\pi} \cos x\, dx$. (c) $\int_0^{\frac{\pi}{2}} \sin x\, dx$.

   (d) $\int_0^{\frac{\pi}{2}} \cos x\, dx$. (e) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\, dx$.

4. From the graphs of $\tan x$, draw your conclusions as to the values of $\frac{d \,.\, \tan x}{dx}$.

   Verify that $\frac{d \,.\, \tan x}{dx} = \sec^2 x = 1 + \tan^2 x$.

   Determine $\frac{d \,.\, \tan x}{dx}$, when $\tan x$ has the value $\sqrt{3}$.

5. On the same axes, construct the graphs of $\sin x$, $\tan x$ and x (in radians) from $x = 0°$ to $x = 45°$, and verify that as x decreases, $\sin x$, $\tan x$ and x become more nearly equal, and that if δx represents a very small angle, $\sin \delta x$, $\tan \delta x$ and δx are practically equal to one another.

   If $\delta x = 1°$, find the error per cent. in writing:

   (i) $\sin \delta x$ for $\tan \delta x$. (ii) $\tan \delta x$ for $\sin \delta x$.

   (iii) δx (in radians) for $\tan \delta x$. (iv) δx (in radians) for $\sin \delta x$.

6. If δx represents a small angle, show that

$$\tan(x + \delta x) - \tan x = \frac{\delta x(1 + \tan^2 x)}{1 - \delta x \tan x}.$$

§ 8. $y^2 = a^2 - x^2$, or $y = \pm\sqrt{a^2 - x^2}$, or $x^2 + y^2 = a^2$.

The graph is the circumference of a circle, the centre of which is the origin and the radius, a (fig. 11).

Take any point P, co-ordinates x, y, on the circumference, and through it draw a tangent PT, cutting the axis of x at T.

Then the gradient of the graph at P is $\frac{QP}{TQ}$.

At P in the figure, the gradient is negative.

Now triangle PQT is similar to triangle OQP; therefore

$$\frac{\text{QP}}{\text{TQ}}=\frac{\text{QO}}{\text{QP}}=\frac{-x}{y}=\frac{-x}{\pm\sqrt{a^2-x^2}}.$$

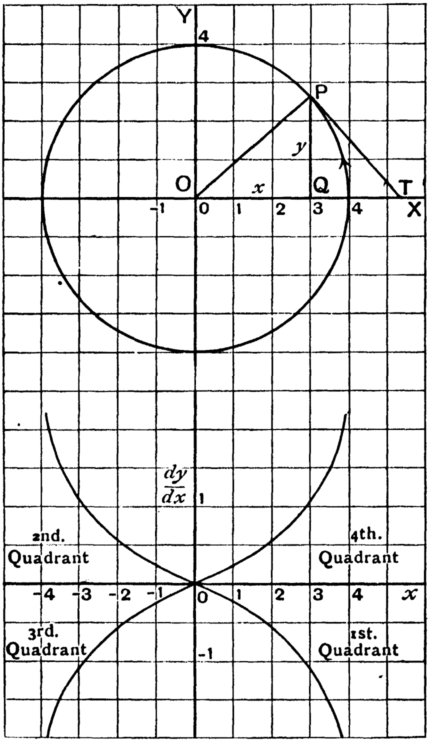


FIG. 11.

Hence, if
$$y=\pm\sqrt{a^2-x^2},$$
$$\frac{dy}{dx}=\frac{-x}{\pm\sqrt{a^2-x^2}} \quad\text{and}\quad \int\frac{-x}{\pm\sqrt{a^2-x^2}}\,dx=\sqrt{a^2-x^2}+c.$$

EXERCISE XXIII (I)

The following exercises refer to § 8.

1. Draw the graph of dy/dx below that of y, and show some corresponding areas and ordinates in the two figures.

2. For what values of x does dy/dx equal 0?

   What are the values of y in these cases, and how do they compare with other values of y?

3. Trace the changes in dy/dx as P moves round the circle (i) clockwise, (ii) counter-clockwise.

4. Imagine the tangent PT to move round the circle with the radius OP. By how many degrees is PT in advance of OP? Contrast the signs in the various quadrants.

5. Discuss the value of dy/dx when x is greater than a.

6. If the centre of the circle is not at the origin but at a point, the co-ordinates of which are $x=b$, $y=c$, then the equation is

$$(x-b)^2+(y-c)^2=a^2, \quad \text{i.e. } y=c\pm\sqrt{a^2-(x-b)^2}.$$

   By the same method as that given in § **8**, show that

$$\frac{dy}{dx}=\frac{-(x-b)}{\sqrt{a^2-(x-b)^2}}.$$

§ 9. The Trigonometrical Functions of 2x.

Draw the graph of $\sin 2x$, measuring x along the horizontal, and $\sin 2x$ along the vertical axis.

Contrast the graph with that of $\sin x$.

Observe that $\sin 2x$ completes half its wave in the horizontal range 0° to 90°, whereas $\sin x$ completes the corresponding half wave in the range 0° to 180°.

As a result, the gradient of $\sin 2x$ is steeper than that of $\sin x$.

In fact, since, as compared with $\sin x$, the values of $\sin 2x$ are attained in half the range, the gradient of $\sin 2x$ is twice that of $\sin x$.

Graph the gradient of $\sin 2x$ with respect to x, and contrast the graph with that of $\cos 2x$.

It is seen that if $y=\sin 2x$, $\dfrac{dy}{dx}=2\cos 2x$.

Similarly, verify that $\dfrac{d}{dx}.\cos 2x=-2\sin 2x$.

It follows that $\int 2\cos 2x\,.\,dx=\sin 2x+c,$

from which $\int \cos 2x\,.\,dx=\frac{1}{2}\sin 2x+c,$

and that $\int \sin 2x\,.\,dx=-\frac{1}{2}\cos 2x+c.$

Generally, $$\frac{d}{dx}\,.\sin ax = a\cos ax$$

and $$\frac{d}{dx}\,.\cos ax = -a\sin ax.$$

Write down the corresponding integrals.

Exercise XXIII (j)

1. Find $\frac{d}{dx}\sin 2x$, when $x = 30°$.

2. Find $\frac{d}{dx}\cos 2x$, when $x = 45°$.

3. Find $\frac{d}{dx}\sin \frac{1}{2}x$, when $x = 60°$.

4. Determine the following integrals:

$$(a)\ \int_0^{\frac{\pi}{2}} \sin 3x. \qquad (b)\ \int_0^{\frac{\pi}{6}} \cos \tfrac{1}{2}x, \qquad (c)\ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin \tfrac{1}{2}x.$$

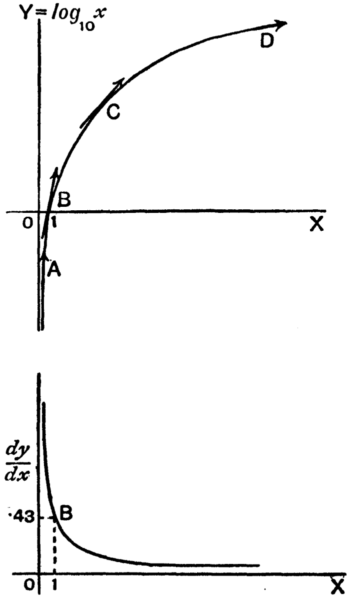


Fig. 12.

§ 10. Differential of the Logarithmic Function.

Plot the graph of the function of x. $\quad y = \log_{10} x$.

Examine the gradient at points A, B, C, D (fig. 12).

At A, the tangent is almost at right angles to the axis of x. The gradient is therefore very great.

The gradient decreases rapidly from A to B, and from B towards D the tangent approaches the horizontal, and the gradient, therefore, the value 0.

These changes recall the graph of $\frac{1}{x}$.

Determine $\frac{dy}{dx}$ by the tangent method, and draw its graph with respect to x.

Find the expression of the graph of $\frac{dy}{dx}$.

$\left(\text{Try } \log\left(\frac{dy}{dx}\right) \text{ against } \log x \text{ or } \frac{dy}{dx} \text{ against } \frac{1}{x}.\right)$

You will find that if $y = \log_{10} x$,

$$\frac{dy}{dx} = \frac{K}{x},$$

where K is the value of $\frac{dy}{dx}$ when $x = 1$.

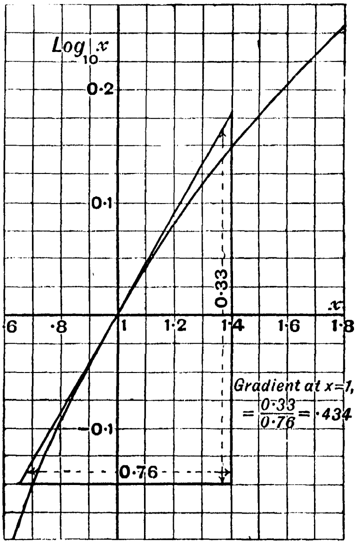


FIG. 13.

The approximate value of K is ·4343 (fig. 13).

It follows that $$\int \frac{\cdot 4343}{x} \,.\, dx = \log_{10} x + c,$$

or $$\int \frac{1}{x} \,.\, dx = \frac{1}{\cdot 4343} \log_{10} x + c,$$

or $$= 2{\cdot}3026 \log_{10} x + c.$$

This last result is specially interesting.

If the ordinates in the $y = \log_{10} x$ graph be multiplied by 2·3026 (practically 2·3), we get a new logarithmic graph, the gradient of which is, of course, $\frac{1}{x}$.

These new logarithms are called Napierian logarithms, and the base is not 10, but a number of which the logarithm to the base 10 is ·4343, viz. 2·718.

The number is denoted by the letter e.

The value of e is the sum of the series:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{etc. without limit.}$$

This series is called the exponential series, and is dealt with in Chapter XXV.

Napierian logarithms are found, then, by multiplying common logarithms by 2·3026.

It follows that (i) If $y = \log_e x$, $\frac{dy}{dx} = \frac{1}{x}$.

(ii) $\int \frac{1}{x} \cdot dx = \log_e x + c$.

Exercise XXIII (k)

1. Find (i) $\log_e 10$. (ii) $\log_e 2$. (iii) $\log_e 0{\cdot}2$.

2. Plot the graph of $\log_e x$, of e^x and of $\frac{1}{e^x}$.

3. Plot the graph of $5e^{-x} . \sin x$, and observe the effect of the factor e^{-x} on the graph of $\sin x$.

   (Measure x in radians from 0 to 4π.)

4. From the graph of $y = e^x$, find $\frac{dy}{dx}$.

   Plot the results, and compare the graph of $\frac{dy}{dx}$ with that of y. Verify that the rate of change of e^x at any value of x is equal to the value of e^x for that value of x. In other words, that the rate of change of e^x is e^x.

5. $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{etc.}$$

   By differentiating each term, find

   $$\frac{d . (e^x)}{dx}.$$

   What do you find?

6. If $$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \text{etc.},$$
where a is a constant, show that
$$\frac{d \, . \, e^{ax}}{dx} = ae^{ax}.$$

The function e^{ax} is such that the ratio of the rates of change at any chosen values is equal to the ratio of those values of the function. Write down the corresponding integral.

7. Place a thermometer in a vessel containing water, the temperature of which is about 20° C. above that of the room.

When the thermometer has attained the temperature of the water, clamp it to a stand, and then remove the vessel of water. Take the readings of the cooling thermometer, say, every minute, and plot them.

(i) Find the rate of fall in temperature.

(ii) Plot the rate of fall and the difference between the temperature of the thermometer and the temperature of the room. Draw your conclusion.

(iii) Plot the logarithm of the excess in temperature of the thermometer above the room, and the time. State the result.

8. Verify graphically that if $y = \log_e(x + 3)$,
$$\frac{dy}{dx} = \frac{1}{x + 3}.$$
Write down the corresponding integral.

9. By writing $\frac{12}{x^2 - 9}$ as the difference between two fractions, show that
$$\int \frac{12 \, . \, dx}{x^2 - 9} = 2 \{ \log_e(x - 3) - \log_e(x + 3) \} = 2 \log_e \frac{x - 3}{x + 3}.$$

10. Find $$\int_1^5 \frac{dx}{x} \quad \text{and} \quad \int_0^6 \frac{dx}{x + 4}.$$

§**11.** The method generally used in finding differential coefficients is illustrated in the following example.

If $y = \tan x$, find $\frac{dy}{dx}$.

Let x take a small increment δx, and let δy be the corresponding change in y.

Then $$y + \delta y = \tan(x + \delta x)$$
and
$$\begin{aligned}
\delta y &= \tan(x + \delta x) - \tan x \\
&= \frac{\tan x + \tan \delta x}{1 - \tan x \tan \delta x} - \tan x \\
&= \frac{\tan x + \tan \delta x - \tan x(1 - \tan x \tan \delta x)}{1 - \tan x \tan \delta x} \\
&= \frac{\tan \delta x + \tan^2 x \tan \delta x}{1 - \tan x \tan \delta x}.
\end{aligned}$$

Since δx is small, we may write δx for $\tan \delta x$.

Hence $$\delta y = \frac{\delta x + \delta x \tan^2 x}{1 - \delta x \tan x},$$

and, dividing by δx, $$\frac{\delta y}{\delta x} = \frac{1 + \tan^2 x}{1 - \delta x \tan x}.$$

Let δy and δx diminish indefinitely. Then the quotient approaches a particular value, which it reaches when each term of the ratio $\frac{\delta y}{\delta x}$ differs by no measurable amount from nought. As already stated, this value is denoted by the symbol $\frac{dy}{dx}$. The term $\delta x \tan x$ in the denominator of the right side of the equation, however, becomes smaller and smaller until it differs by no measurable amount from nought, that is, finally vanishes, and the equation becomes

$$\frac{dy}{dx} = \frac{1 + \tan^2 x}{1 - 0} = 1 + \tan^2 x = \sec^2 x.$$

Exercise XXIII (l)

1. Find the rate at which the function $y = \tan x$ changes at the value $x = 60°$.

2. Show that the tangent drawn to the curve $y = \tan x$, at the point at which $x = 0°$, makes an angle of $45°$ with the axis of x.

3. Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$ and $\int_{30°}^{60°} \tan^2 x\, dx$.

4. Show that $\int \tan^2 x \,.\, dx = \tan x - x + c$.

5. Find $\displaystyle\int_0^{60^\circ} \frac{dx}{\cos^2 x}$ and $\displaystyle\int_{45^\circ}^{60^\circ} \frac{dx}{1-\sin^2 x}$.

6. Determine $\displaystyle\int (\sec^2 x + \tan^2 x)\,dx$.

CHAPTER XXIV

APPLICATIONS OF THE CALCULUS, AND EXERCISES

§ 1. Maximum and Minimum Values.

Referring to your practical work on the functions, $ax^2 + bx + c$, $\sin x$, $\cos x$, etc., you have found that at the points of maximum and minimum value, the gradient is 0; that is, the differential coefficient is 0.

This fact enables us to determine the value of x for which the function is a maximum or a minimum.

EXAMPLE i.

$$y = ax^2 + bx + c,$$
$$\frac{dy}{dx} = 2ax + b.$$

For minimum value, $\dfrac{dy}{dx} = 0.$

Hence $2ax + b = 0,$

from which $x = -\dfrac{b}{2a}$

and
$$y = \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{-b^2}{4a} + c$$
$$= \frac{4ac - b^2}{4a}.$$

This agrees with the result on page 180.

EXAMPLE ii.

$$y = \sin x,$$
$$\frac{dy}{dx} = \cos x.$$

For a maximum or minimum, $\cos x = 0$,

i.e. when x is 90° or 270°.

Hence $\sin x$ is a maximum or a minimum when x is 90° or 270°.

§2. The Trajectory of a Projectile.

The equation given on page 221, for the path of a projectile, can be more easily established as follows:

$$y = ax^2 + bx; \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

therefore

$$\frac{dy}{dx} = 2ax + b. \dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$$

At the point $x = 0$, $\frac{dy}{dx}$ is the elevation of the gun, and substituting 0 for x, we have

$$\frac{dy}{dx} = b. \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(iii)}$$

That is, the value of the tangent of the angle the gun makes with the horizontal is b,

$$\text{i.e.} \quad \tan e = b. \dots\dots\dots\dots\dots\dots\dots\dots \text{(iv)}$$

Again, the maximum altitude is $\frac{-b^2}{4a}$, and in terms of the velocity is $\frac{V^2 \sin^2 e}{2g}$.

Hence

$$-\frac{b^2}{4a} = \frac{V^2 \sin^2 e}{2g},$$

from which

$$a = -\frac{b^2 g}{2V^2 \sin^2 e}$$

$$= -\frac{g}{2V^2}\frac{\tan^2 e}{\sin^2 e}, \text{ from (iv)},$$

$$= -\frac{g}{2V^2 \cos^2 e}$$

$$= -\frac{g}{2V^2}\sec^2 e.$$

Equation (i) then becomes

$$y = -\frac{gx^2}{2V^2}\sec^2 e + x \tan e.$$

Exercises XXIV (a)

1. If the number of revolutions (N) made by a rotating wheel when rising in speed is given by the equation $N = \frac{1}{4}t^2$, where t is the time in seconds after the instant of starting, find the speed after 2, 10, 12, 20 and 30 seconds. That is, find $\frac{d.N}{dt}$.

2. The speed of a wheel is taken at intervals of 5 seconds reckoned from the time of starting, and was found to be as follows:

Time (secs.) - -	0	5	10	15	20
Speed (revs. p. min.)	0	30	60	90	120

Find the total number of revolutions made in 30 seconds.

3. The speed of a moving body at different instants is given in the table:

Speed (ft. per sec.)	256	224	192	160	128	etc.
Time (secs.) - -	0	1	2	3	4	

Find the distance covered in coming to rest.

4. Turn to Exercise XVII (A), No. 14. Find the angle of elevation or quadrant angle of the gun.

5. Find the maximum value of $1+6x-x^3$, for positive values of x, and state the corresponding value of x.

6. Arrange the equation $y=\frac{-gx^2}{2v^2}\sec^2 e+x\tan e$ in a convenient form for finding e; then find the angle of elevation (e) of a gun in order that an object at an altitude (y) 1000 feet and at a horizontal distance (x ft.) of 2000 yards may be hit, the muzzle velocity (v) being 2500 ft. per second. Account for the two answers.

7. Turn back to Exercise XVII (A), page 181, and solve exercises, from 7 to the end, by differentiating the expressions.

8. Refer to the graph of $y=\pm\sqrt{a^2-x^2}$ on page 292, and to that of $y=c\pm\sqrt{a^2-(x-b)^2}$, Exercise XXIII (I), No. 6, and check the maximum and minimum points by calculation.

9. Find the dimensions of the rectangle which has a maximum area for a given perimeter p.

10. Find the dimensions of a cylinder such that for a combined circumference and length of 9 feet, the volume may be a maximum.

§3. Magnetic Force.

(1) The force of a magnetic pole, at a point, varies inversely as the square of the distance between the pole and the point.

If the strength of the magnetic pole is m units, it will exert on another magnetic pole having unit strength, placed at a distance x, a force of $\frac{m}{x^2}$ units.

If the poles are of like kind, say both North, the force is a repulsion; if of unlike kind, an attraction.

In order to illustrate the problem, set up a magnet in a vertical position over a bowl of water. Float in the water a small piece of cork, through which is pushed a small magnetised sewing needle, the end projecting upwards being such that the needle is repelled by the magnet (fig. 1).

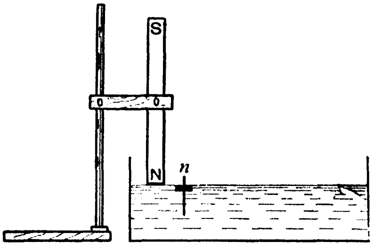


FIG. 1.

(i) Construct a graph showing the force at different distances from the pole of the magnet, when m is, say, 10 units.

(ii) Determine the rate of change of force with distance.

(2) Imagine the unit pole to be free to move, then the work done when the unit pole is repelled from a distance x_1 to a distance x_2, is the product of the force (which varies according to the inverse square law stated above) and the distance through which the unit pole moves. That is, it is the integral of $\frac{m}{x^2}dx$ from $x=x_1$ to $x=x_2$ (see fig. 2).

(i) Find this integral, and so obtain a general expression for the product.

The result for the limits stated is $m\left(\frac{1}{x_1}-\frac{1}{x_2}\right)$.

Now suppose x_2 to increase in magnitude until it becomes infinitely great; then, when the unit pole is at an infinite distance

from the other magnetic pole, $\frac{1}{x_2}$ differs from 0 by no measurable amount, and the above expression for the work done finally becomes $\frac{m}{x_1}$. This value is called the magnetic potential at a point a distance x_1 from a magnetic pole of strength m.

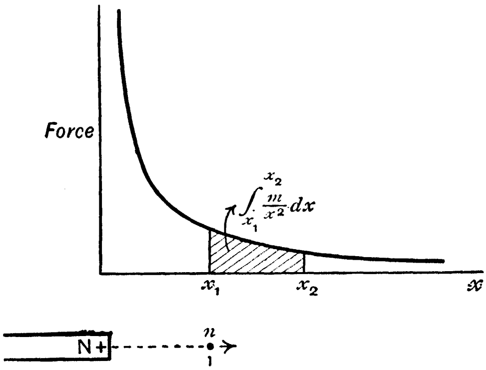


FIG. 2.

It is the measure of the work done if the unit pole were repelled from the distance x_1 to an infinite distance, and therefore represents the potential energy at the point.

(ii) Draw a graph showing potential and distance.

(iii) Find the potential at distances (a), 10 cms., (b), 25 cms. from a magnetic pole of strength 50 units.

(iv) What is the difference between the potentials in Ex. (iii)?

(v) At what rate does the force of the pole in Ex. (iii) change with distance?

(vi) At what rate does the potential due to a magnetic pole change with distance? Note carefully the result.

(3) Consider next the effect of two unlike poles of equal strength (a magnet has usually two such poles) at a distance of,

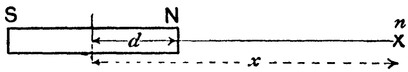


FIG. 3.

say, 5 cms. apart, on a unit N pole placed in the line joining the two poles, produced (see fig. 3).

One pole (N) repels the unit pole, and the other (S) attracts it. The resultant force is the difference between the forces.

(i) Draw the graph of repulsion with distance, and on the same axes that of attraction with distance. What do you notice about the two graphs? (Take each pole to be of strength 100 units.)

(ii) Subtract the ordinates, and so obtain the graph of the force of the magnet. Observe that, as the distance increases, the distance the poles are apart becomes less and less significant.

(iii) Taking the resultant graph, and reckoning distance from the centre of the magnet, see if you can, by the logarithmic method, find the law connecting force and distance.

(iv) From what point of the graph obtained in Ex. (ii) does the formula $\frac{4md}{x^3}$ agree closely?

(v) Find the potential at a distance x from a short magnet, given that:

$$\text{Potential} = \int F\,dx$$

and
$$F = \frac{4md}{x^3}.$$

(vi) Taking the magnetic potential due to a magnet, at a point in the straight line joining its poles, produced, as being equal to the difference of the potential due to each pole, determine the true expression for the potential at a distance x from the centre of the magnet. Find the error per cent. made by using the formula obtained in (v) when $d = 4$ and $x = 12$.

§ **4.** When a gas expands at constant temperature, the relation between pressure and volume is given by the equation

$$pv = K, \text{ where K is a constant.}$$

It follows that $p = K/v$.

The work done when the gas expands from a volume v_1 to a volume v_2, is the product of pressure and volume between these limits—the pressure changing according to the law, $pv = K$.

$$\text{Hence, Work} = \int_{v_1}^{v_2} p\,dv = \int_{v_1}^{v_2} \frac{K}{v}\,dv = K\Big[\log_e v\Big]_{v_1}^{v_2}$$
$$= K(\log_e v_2 - \log_e v_1)$$
$$= K\log_e \frac{v_2}{v_1}.$$

K is found from any simultaneous values of p and v.

§ **5.** When heat is not allowed to enter or to escape from the gas, the change is said to be **adiabatic,** and the relation between pressure and volume is: $pv^s = \mathrm{K}$, from which $p = \frac{\mathrm{K}}{v^s}$.

The work done when the gas expands from v_1 to v_2 is:

$$\int_{v_1}^{v_2} p\,dv = \int_{v_1}^{v_2} \frac{\mathrm{K}}{v^s}\,dv = \int_{v_1}^{v_2} \mathrm{K}v^{-s}\,dv = \mathrm{K}\Big[\frac{v^{(1-s)}}{1-s}\Big]_{v_1}^{v_2}$$

$$= \frac{\mathrm{K}}{1-s}\left(v_2^{(1-s)} - v_1^{(1-s)}\right)$$

or

$$= \frac{\mathrm{K}}{s-1}\left(\frac{1}{v_1^{(s-1)}} - \frac{1}{v_2^{(s-1)}}\right).$$

For most gases, s equals 1·41.

i. Find the work done when a gas at a pressure of 100 lbs. per sq. inch, and of volume 1·5 cub. feet, expands to a volume 3·5 cub. feet, (i) at constant temperature, (ii) adiabatically.

§ **6.** Fill a burette with coloured water, fix it vertically in a stand, and open the tap so that the level of the liquid falls at a convenient rate

Using a seconds' watch or clock, take the time the surface has the readings 0, 5, 10, etc.

Plot a graph representing the relation between the position of the surface, and time.

Find the gradient, i.e. the rate of change of the position, and therefore the velocity of the surface, at different points.

Measure the distance of, say, the last graduation mark above the outlet, in order that the total length of the column of liquid (called the head) corresponding to each position of the surface may be determined.

Plot the graph representing the relation between the rate at which the surface falls, and the head.

Using the method of logarithms, find as closely as possible how the velocity of the surface, which, in this case, is also the velocity at which the liquid flows out, depends upon the head.

§ **7.** Draw out the tube of a glass funnel to a fine bore. Mount the funnel vertically in a stand with a scale made by pasting a piece of squared paper on cardboard, and cutting the cardboard so that when mounted, with one set of lines vertical, the prepared edge is in contact with the side of the funnel (fig. 4).

The vertical fall of the surface can then be readily measured. Place a graduated cylinder to receive the liquid.

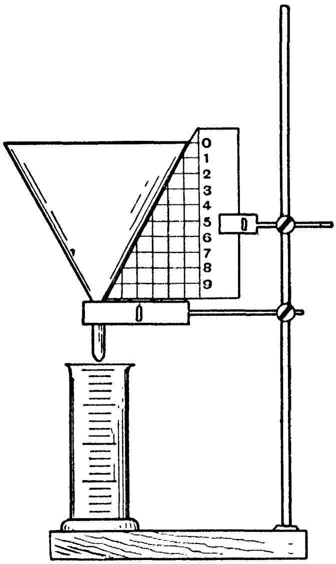


FIG. 4.

Determine:

(i) The time at which the surface is at certain levels.

(ii) The velocity of descent of the surface.

(iii) The rate at which the water flows out.

(iv) How (iii) depends upon the head.

§ **8.** Take a tin of hot water, observe the temperature at convenient intervals of time, and draw the curve of cooling. Predict the time at which the water will have a certain temperature. See if your prediction is correct. Identify the curve, if you can.

§ **9.** Use in an inverted position a round-bottomed flask, with an air hole blown in the bottom, and fitted with a bung through which passes a straight glass tube, to obtain graphs like those of § **7.**

Construct a graph showing the rate at which the surface descends.

§10. Water flows out of a tank, at a rate at any instant proportional to the depth of the water remaining in the tank at that instant.

Illustrate this statement graphically. Observe that the rate is a function of the depth and that the depth is a function of time. What is the form of the latter function? Show by differentiation that it satisfies the condition.

§11. If $y = ar^x$, where a and r are constants, show by writing, say, l for $\log_e r$, that dy/dx is alr^x, i.e. $a \log_e r \cdot r^x$.

Find also $\int ar^x dx$, and contrast it with the sum of $(x+1)$ terms of a G.P. Account for the difference.

§12. If a fly-wheel is available, determine by tachometer the speed at noted instants when the wheel is running down, and from the graph of the results find the total revolutions made in a stated interval of time, or the total revolutions made in running down to rest. If a revolution counter also is used, the results can be ckecked.

§13. The figure illustrates a piece of mechanism. D is a disc of diameter 18 inches, pivoted eccentrically at O, a point 6 inches from the centre. AB is a rod which is moved up and down when D rotates.

(i) Construct a graph showing the position of A, as D rotates through a complete cycle. (Draw a number of lines radiating from O at equal angles, as indicated in the figure.)

(ii) If D rotates at a uniform speed, it turns through equal angles in equal times. Find from graph (i) the rate of change of the position of A when the disc makes 60 revs. per minute (i.e., find the velocity of A when passing through its various positions). Graph the results.

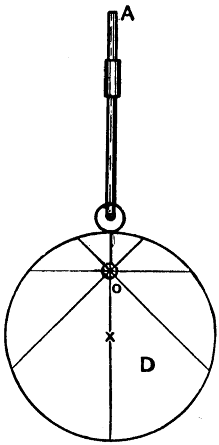


FIG. 5.

§14. **Induced Electro-motive Force.**

When a loop of wire rotates in a uniform magnetic field, the axis of rotation being at right angles to the direction of the lines of force of the magnetic field, then the number of lines threading

the loop at any instant varies as the cosine of the angle turned through by the loop from the position in which the maximum number is threaded; that is, from the position in which the plane of the loop is at right angles to the lines.

Draw a graph showing the number of lines threaded and the angle turned through by the loop.

The voltage, or electro-motive force induced in the loop, varies as the rate at which the number of lines of magnetic force in the loop changes, and its direction is such that it opposes the change.

How does the rate of change vary with the angle turned through by the loop, the rotation being uniform?

Construct a graph illustrating your answer.

The graph showing how the voltage varies with the angle is precisely opposite to this graph. Construct it.

If $y = \mathrm{F}\cos ax$ and $\mathrm{V} = -k\dfrac{dy}{dx}$, find V.

§**15.** The figure shows a mechanism for converting circular motion into straight-line reciprocating motion. As the disc D rotates about its centre, the pin P, which is fixed to it, causes the slide S to move backwards and forwards.

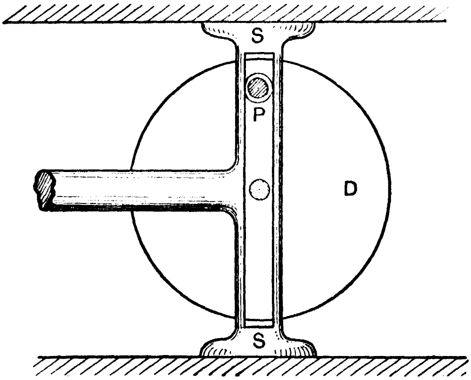


Fig. 6.

If P is situated 10 inches from the centre of the disc, draw a graph showing the displacement of the slide for various angles turned through by the disc.

If the disc rotates uniformly at 60 revs. per min., draw a graph

showing the velocity of the slide when passing through any position, and another showing the velocity and time.

Now construct a graph showing the rate of change of velocity, i.e. acceleration.

What is the maximum velocity of the slide?

§ **16.** A point moves so that its position S is given by the formula $S = at^3 + bt^2 + c$, where t is the time and a, b and c are constants. Find its velocity, and its acceleration in terms of a, b, c and t.

When is the velocity a minimum?

Apply the results to the formula $t^3 - 27t^2 + 8$.

§ **17.** To find the area bounded by the parabolic graph $y = x^2$ (see p. 197).

Regard the graph as the differential of another graph. The other graph, you will remember, is obtained by integration; that is, by finding the area bounded by the parabolic graph.

The formula $\int x^{n-1}\,dx = \frac{1}{n} x^n + c$ can be used.

$$\begin{aligned} \int x^2\,dx &= \frac{1}{2+1} x^{2+1} + c \\ &= \tfrac{1}{3}x^3, \text{ when } c = 0, \\ &= \tfrac{1}{3}xy, \text{ when } y \text{ is substituted for } x^2. \end{aligned}$$

That is, the area bounded by the graph $y = x^2$, the axis of x and the end ordinate y, is $\frac{1}{3}xy$.

(i) Show that the area bounded by the graph, an abscissa, and the axis of y is $\frac{2}{3}xy$.

§ **18. The Circumference and Area of a Circle.**

(1) Consider any sector.

Let the angle contained by the two radii be θ radians, the radius R and the arc y.

Then $$y = R\theta \quad \text{(i)}$$

Now let the angle be increased by a small amount $\delta\theta$, and let the corresponding change in y be δy.

Then $$y + \delta y = R(\theta + \delta\theta) \quad \text{(ii)}$$

Subtracting equation (i) from equation (ii), we have

$$\delta y = R\delta\theta \quad \text{(the increment in the arc),}$$

and therefore $$\frac{\delta y}{\delta\theta} = R.$$

When δy and $\delta\theta$ are diminished indefinitely, we obtain

$$\frac{dy}{d\theta}=\mathrm{R},$$

i.e. the rate of change of the arc with respect to the angle it subtends at the centre is equal to the radius (a constant).

It follows that $\int \mathrm{R}\, d\theta = y$ (the arc), and this is true for all values of θ.

Therefore, Circumference of circle $= \int_0^{2\pi} \mathrm{R}\, d\theta$

$$= \mathrm{R}\Big[\theta\Big]_0^{2\pi}$$

$$= 2\pi\mathrm{R}.$$

(2) Similarly, since the area of a small sector of angle $\delta\theta$ is $\frac{1}{2}\mathrm{R}^2\delta\theta$,

$$\text{Area of circle} = \int_0^{2\pi} \tfrac{1}{2}\mathrm{R}^2\, d\theta$$

$$= \tfrac{1}{2}\mathrm{R}\Big[\theta\Big]_0^{2\pi}$$

$$= \pi\mathrm{R}^2.$$

(i) Show, as in (1), that the rate of change of area of a sector of a circle with respect to the angle is $\frac{1}{2}\mathrm{R}^2$.

§19. Average Value of the Sine Function Ordinates from 0 to $\frac{\pi}{2}$.

The problem is really, "What is the height of the rectangle on the base 0 to $\frac{\pi}{2}$, whose area is equal to that bounded by the sine graph from 0 to $\frac{\pi}{2}$?"

$$\text{Area bounded by sine graph} = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \Big[-\cos x\Big]_0^{\frac{\pi}{2}}$$

$$= \Big[-\cos\frac{\pi}{2} - (-\cos 0)\Big]$$

$$= [0-(-1)]$$

$$= 1;$$

$\therefore$ **area of rectangle $= 1$.**

Since base of rectangle $=\frac{\pi}{2}$, height of rectangle $=\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi}$,

i.e. average value $=\frac{2}{\pi}=\cdot 636.$

(i) Find the average value of $\cos x$ from $x=0$ to $x=\frac{\pi}{2}$.

(ii) Find the average value of $\sin x$ from $x=0$ to $x=\pi$.

(iii) Find the average value of $\cos x$ from $x=0$ to $x=\pi$.

(iv) Of what angle is the sine equal to the average value of the sine from 0 to $\frac{\pi}{2}$?

§ **20.** The average of the square of the ordinates of the sine curve is useful in Engineering.

(i) It is readily deduced as follows:

Since the ordinates of the sine curve have the same value from 0 to $\frac{\pi}{2}$ as those of the cosine curve from $\frac{\pi}{2}$ to 0, the squares of the corresponding ordinates must be the same, and the average of the squares of the ordinates of these two curves must be the same

(e.g. $\sin 30^\circ = \cos(90^\circ - 30^\circ)$

and $\sin^2 30^\circ = \cos^2(90^\circ - 30^\circ)$,

see fig. 7).

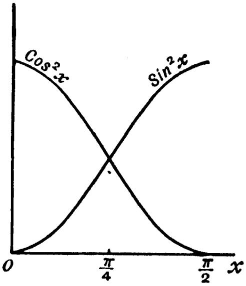


FIG. 7.

Now, $\sin^2 x + \cos^2 x = 1$,

and this is true for all values of $\sin^2 x$ and $\cos^2 x$, and true, therefore, for their average values, which we have seen are equal.

Hence, $2\sin^2 x = 1$ and $2\cos^2 x = 1$,

where x denotes the angle at which the squares of $\sin x$ and of $\cos x$ have their average value.

Therefore, $\sin^2 x = \frac{1}{2}$, and also $\cos^2 x = \frac{1}{2}$,

i.e. the average of the squares of the ordinates of the sine or cosine curve is $\frac{1}{2}$.

The root of the average (or mean) square is $\sqrt{\frac{1}{2}} = 0{\cdot}707$.

For what angle have the sine and cosine this value?

(ii) It may be calculated as follows:

$$\text{Average value of } \sin^2 x = \frac{\int_0^{2\pi} \sin^2 x\, dx}{2\pi}. \qquad \text{(i)}$$

Now, $$\sin^2 x = \frac{1 - \cos 2x}{2} = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\,;$$

therefore

$$\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx = \tfrac{1}{2}\int_0^{2\pi} dx - \tfrac{1}{2}\int_0^{2\pi} \cos 2x\,dx$$
$$= \pi - \tfrac{1}{4}\Big[\sin 2x\Big]_0^{2\pi}$$
$$= \pi - 0$$
$$= \pi.$$

Hence, equation (i) becomes:

$$\text{Average value of } \sin^2 x = \frac{\pi}{2\pi}$$
$$= \tfrac{1}{2}.$$

Exercise.—Show that in (ii) the same result is obtained when the limits of the integration are $\frac{\pi}{2}$ and 0 and the divisor in equation (i), $\frac{\pi}{2}$.

Note.—It is an interesting exercise to complete the graph of $\cos^2 x$ for the range $x = 0$ to 2π, and to show, by drawing a new horizontal axis at $y = \frac{1}{2}$, that $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ and $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

§21. Volume of a Paraboloid.

If the parabola $y^2 = kx$ revolves about the axis of x, a solid called a paraboloid is generated.

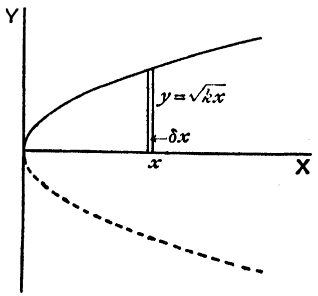


Fig. 8.

The strip of thickness δx, at a distance x, generates a disc

of volume $\pi y^2 \delta x$, or $\pi kx \delta x$, which shows how the volume changes with respect to x.

The volume of the whole paraboloid is therefore $\int \pi kx\, dx$

$$= \tfrac{1}{2}\pi kx^2.$$

(i) Show that $\frac{1}{2}\pi kx^2$ is half the product of the base and altitude of the paraboloid.

(ii) A solid wooden cylinder is hollowed out so that the interior is a hollow paraboloid. What fraction of the solid cylinder remains?

§ **22.** To find the area of the figure bounded by the curve described by a point on the circumference of a circle and a straight line upon which the circle rolls.

Let P be the point on the circumference, and let its initial position be that of the upper extremity of the diameter at right angles to the straight line (fig. 9).

Let P_1 be the position reached by P, after the circle has turned through an angle θ.

Then the centre moves forward a distance $r\theta$, where r is the radius, and P moves a horizontal distance $r \sin \theta$ from the vertical diameter.

Let y be the vertical ordinate and x the abscissa of any point P on the curve. Then

$$\text{(i)}\ y = r(1 + \cos\theta) \quad \text{and} \quad \text{(ii)}\ x = r(\theta + \sin\theta).$$

From (ii), $$x = r(\theta + \sin\theta),$$

$$dx/d\theta = r(1 + \cos\theta). \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

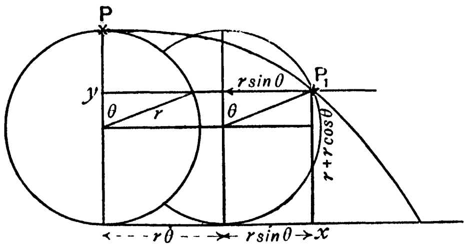


FIG. 9.

Consider the area bounded by the curve traced by P when the circle turns through π radians (i.e. half a revolution).

For this we require the integral of $y\,dx$ between the limits $x=0$ and $x=r\pi$, the corresponding limits for θ being 0 and π.

$$\int_0^{r\pi} y\,dx = \int_0^{\pi} y \cdot \frac{dx}{d\theta} \cdot d\theta.^* \quad \text{See (iv) below.}$$

$$= r^2\int_0^{\pi}(1+\cos\theta)^2 d\theta \text{ ....from (i) and (iii)}$$

$$= r^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= r^2\left(\int_0^{\pi}d\theta + 2\int_0^{\pi}\cos\theta\,d\theta + \int_0^{\pi}\cos^2\theta\,d\theta\right)$$

$$= r^2\Big[\theta + 2\sin\theta + \int\frac{1+\cos 2\theta}{2}\cdot d\theta\Big]_0^{\pi}$$

$$= r^2\Big[\theta + 2\sin\theta + \tfrac{1}{2}\int d\theta + \tfrac{1}{2}\int\cos 2\theta\,d\theta\Big]_0^{\pi}$$

$$= r^2\Big[\theta + 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\Big]_0^{\pi}$$

$$= r^2(\pi + 0 + \tfrac{1}{2}\pi + 0)$$

$$= \tfrac{3}{2}\pi r^2.$$

For the full curve, the area is $3\pi r^2$; that is, three times the area of the rolling circle.

The curve is called a **Cycloid.**

(i) Examine the equation $y=r(1+\cos\theta)$, when θ is 0°, 45°, 90°, 120°, 180°.

(ii) Find the area bounded by the graph, the end ordinates and the portion of the axis of x for the range corresponding to a rotation from the position of P in the figure, through an angle of (i) 45°, (ii) 90°.

(iii) By reference to a graph, show that $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$.

(iv) If $y=az^2$, and $z=bx+c$, verify that $\dfrac{dy}{dx}=\dfrac{dy}{dz}\times\dfrac{dz}{dx}$, and that if $y'=\dfrac{dy}{dx}$, $\int y'\,dx = \int\left(y' \times \dfrac{dx}{dz}\right)dz$.

Repeat with (i) $z=\dfrac{b}{x}+c$, (ii) $z=bx^2+c$, (iii) $z=\log_e x$, (iv) $z=e^x$, (v) $z=\sin x$; and also with $y=az^n$.

* *Note carefully the method of changing the variable from x to θ.*

(v) From equation (i), find $dy/d\theta$; from (ii), $dx/d\theta$. Now divide $dy/d\theta$ by $dx/d\theta$, and so find dy/dx, the gradient of the graph.

(vi) Examine the gradient at points corresponding to $\theta = 0°$, $45°$, $90°$, $120°$, $180°$. (Transform the expression for the gradient into a function of $\frac{1}{2}\theta$ when substituting $180°$.)

(vii) Trace the changes in the gradient as θ increases from $120°$ to $180°$.

§ **23.** Prove that the volume of a cone is $\frac{1}{3}$ the product of the base and the altitude.

A hollow cone is gradually filled with water.

Show that the rate at which the volume changes with respect to the depth is $\pi k^2(h-x)^2$, where x is the depth, h the height of the cone and k the ratio of the radius of the base to the altitude of the cone. For what value of x is the rate a minimum?

§ **24. Volume of a Sphere.**

The ordinate (y) of the semicircle, shown in the figure, at a distance x from the end of the diameter, is given by the equation

$$y^2 = x(\mathrm{D} - x)$$
$$= \mathrm{D}x - x^2.$$

If the semicircle revolves about the diameter it generates a sphere, and the ordinate y, a circle of area πy^2. Consider a very thin disc or zone of the sphere of thickness δx, at this position. Its volume is $\pi y^2\,\delta x$ or $\pi(\mathrm{D}x - x^2)\,\delta x$.

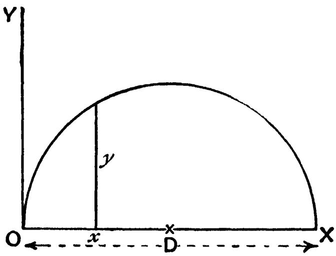


FIG. 10.

$$\therefore \text{ volume of the whole sphere} = \int_0^{\mathrm{D}} \pi(\mathrm{D}x - x^2)\,dx$$
$$= \pi\mathrm{D}\int_0^{\mathrm{D}} x\,dx - \pi\int_0^{\mathrm{D}} x^2 dx$$
$$= \pi\mathrm{D}\left[\frac{x^2}{2}\right]_0^{\mathrm{D}} - \pi\left[\frac{x^3}{3}\right]_0^{\mathrm{D}}$$
$$= \frac{\pi\mathrm{D}^3}{2} - \frac{\pi\mathrm{D}^3}{3}$$
$$= \frac{\pi\mathrm{D}^3}{6} \text{ or } \tfrac{4}{3}\pi\mathrm{R}^3.$$

(Compare this with the statements on page 203.)

Similarly, find the volume of the cap of thickness t.

§ 25. **Surface of a Sphere.**

Consider a thin sector at an angle θ from the diameter of the semicircle (fig 11).

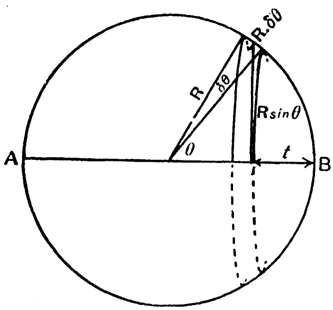


FIG. 11.

The small arc $R\,\delta\theta$ is subtended by a small angle $\delta\theta$ at the centre of the circle. This arc sweeps out a narrow belt of the surface of the sphere generated when the semicircle revolves.

Then the radius of the belt is $R \sin\theta$ and the circumference $2\pi R \sin\theta$.

The area of the narrow belt is

$$2\pi R \sin\theta \times R\,\delta\theta = 2\pi R^2 \sin\theta\,\delta\theta.$$

Area of the surface of the sphere $= \int_0^\pi 2\pi R^2 \sin\theta\, d\theta$

(The limits 0 to π include the whole semicircle, and therefore the whole sphere)

$$= 2\pi R^2 \Big[-\cos\theta\Big]_0^\pi \dagger$$

$$= -2\pi R^2(\cos\pi - \cos 0)$$

$$= -2\pi R^2(-1-1)$$

$$= 4\pi R^2.$$

If the integration is made between the limits 0 and θ such that $\cos\theta = \dfrac{R-t}{R}$, we obtain the curved surface of a spherical cap of thickness t (see the figure).

From the line marked †, we have:

Curved surface of spherical cap $= 2\pi R^2 \Big[-\cos\theta\Big]_0^\theta$

$$= -2\pi R^2(\cos\theta - \cos 0)$$

$$= -2\pi R^2\left(\frac{R-t}{R} - 1\right)$$

$$= -2\pi R^2\left(-\frac{t}{R}\right)$$

$$= 2\pi R t.$$

Exercise XXIV (b)

1. Find the area of a belt or curved surface of a zone. Mark the limits adopted.

2. Using a pair of compasses, a circle is described, as shown in the figure, on a sphere. Without altering the stretch of the compasses, a circle is described on a sheet of paper. Prove that the area of the circle on the paper and the area of the curved surface of the cap of the sphere, limited by the circle drawn on it, are equal.

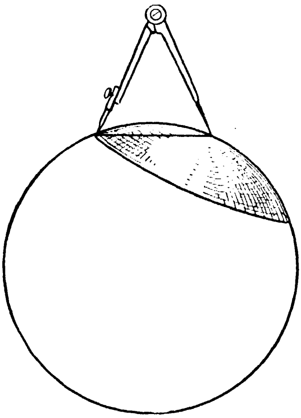


Fig. 12.

What conclusion do you draw regarding caps of spheres differing in radius, marked off in this way?

Prove this without reference to the circle on the plane sheet.

3. Equal chords are drawn, one in each of two unequal circles of radii R_1 and R_2.

If t_1 and t_2 are respectively the projections of these chords on the diameters drawn from one end of each chord, prove that $R_1 t_1 = R_2 t_2$.

Also, if l is the length of the chord, show that $t_1 = \frac{l^2}{2R_1}$.

4. A sphere is enclosed in a cylinder having the same dimensions. Show that the areas of the curved surfaces between two right sections made through both the sphere and the cylinder are equal.

5. Calculate the area of the zones of the earth, viz:

Frigid zones or caps,	limiting latitudes	$66\frac{1}{2}°$ and $90°$ N. and S.
Temperate zones,	,, ,,	$23\frac{1}{2}°$ and $66\frac{1}{2}°$ N. and S.
Torrid zone,	,, ,,	$23\frac{1}{2}°$ N. and $23\frac{1}{2}°$ S.

6. The figure is the section of a double convex lens.

It may be considered as consisting of caps of two spheres of equal radii.

If the diameter of the lens is $2c$ and the thickness $2t$, find the radius of the surfaces and the volume of the lens.

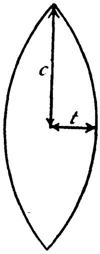


FIG. 13.

7. Obtain a lens like that described in the above question. Measure its diameter and thickness, and from these dimensions calculate its volume.

Verify your result by water displacement.

8. Water is poured into a sphere. Find an expression for the rate at which the volume of water in the sphere increases with the rise (t) of the surface. Show that the rate is a maximum when $t = \mathrm{R}$, R being the radius of the sphere.

9. Draw the graph of $y = \sqrt{a^2 - x^2}$ from $x = 0$ to $x = a$ Draw any ordinate for a value of x between 0 and a, and show that the area between this ordinate and that at $x = 0$ is equal to $\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$. (The area consists of a sector and a triangle.)

10. From 9, show that

$$\text{(i)} \quad \int \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

$$\text{and} \quad \text{(ii)} \quad \int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{4}.$$

CHAPTER XXV

PERMUTATIONS AND COMBINATIONS, EXPANSION OF BINOMIALS, EXPONENTIAL SERIES

§1. Permutations and Combinations.

The subject of Permutations and Combinations is concerned with the number of ways in which a number of things can be

grouped together, arranged, or selected from a given set of things.

The distinction between permutations and combinations is as follows:

In **Permutations**, a different order of the same things is regarded as a different arrangement; whereas, in **Combinations**, no regard is paid to order, but only to the things which constitute a group.

For example, if we had two counters, one red and the other blue, then, placing them together, say one on top of the other, only one combination is possible, the position of the red counter with respect to the blue being of no consequence. On the other hand, there are two permutations possible, namely, one in which the red counter is above the blue, and the other in which the blue is above the red.

§ **2. Permutations.**

(1) To find the number of permutations of n things, taken r at a time.

This number is usually denoted by ${}_nP_r$.

The n different things are conveniently represented by the letters of the alphabet, but without restriction as to number.

Let a, b, c, d, etc., represent some of the different things.

Consider the selection of:

(i) *One letter.*

Since there are n different letters, the number of ways in which one can be selected is n;

$$\text{i.e.} \quad {}_nP_1 = n.$$

(ii) *Two letters.*

One letter can be selected in n ways. Consider only one of these selections, say the letter a. There are now $(n-1)$ letters left. A letter to be placed with a can be chosen from the remaining $(n-1)$ letters in $(n-1)$ ways. This is true for each of the n letters selected first, and, therefore, for all the n letters selected first there are $n(n-1)$ ways of selecting the second, and, therefore, $n(n-1)$ ways of selecting two things from n;

$$\text{i.e.} \quad {}_nP_2 = n(n-1).$$

Note.—In these permutations, any two letters, say a and b, will occur twice; once when a was first selected and b from the remaining $(n-1)$ letters, and once when b was first selected and a from the remaining $(n-1)$ letters.

Also, that the first letter is always chosen from n letters, i.e. the letters are put back after each selection of 2 is made.

(iii) *Three letters.*

Two letters can be selected in $n(n-1)$ ways. After each selection of two, $(n-2)$ letters will remain. Another letter can be selected from the $(n-2)$ letters in $(n-2)$ ways, and since it is true for every one of the $n(n-2)$ selections of two letters, three letters can be chosen in $n(n-1)(n-2)$ ways;

$$\text{i.e.} \quad {}_nP_3 = n(n-1)(n-2).$$

In this case, the same letters, say a, b, c, will occur no less than 6 times, the groups being as follows:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

(iv) Similarly, ${}_nP_4 = n(n-1)(n-2)(n-3)$,

$${}_nP_5 = n(n-1)(n-2)(n-3)(n-4), \quad \text{etc.}$$

Observe that in each case the last bracket consists of n minus one less than the number of letters taken at a time.

Hence, when r letters are taken at a time, the expression representing ${}_nP_r$ will consist of a similar product, the last bracket of which will contain n minus $\overline{r-1}$;

$$\text{i.e.} \quad {}_nP_r = n(n-1)(n-2)(n-3)(n-4)\ldots(n-\overline{r-1});$$

$$\text{i.e.} \quad {}_nP_r = n(n-1)(n-2)(n-3)(n-4)\ldots(n-r+1). \quad \ldots\text{(i)}$$

(2) *Special case.*

To find the number of permutations of n things taken n at a time.

In this case $r=n$.

$$\text{Hence} \quad {}_nP_n = n(n-1)(n-2)(n-3)\ldots(n-n+1),$$

$${}_nP_n = n(n-1)(n-2)(n-3)\ldots(1).$$

This product, the factors of which run down to 1, is called *Factorial* n, and is written as either $\lfloor n$ or $n!$,

$$\text{i.e.} \quad {}_nP_n = n!. \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

In order to extend product (i) to factorial n, it would be necessary to multiply it by a product the factors of which range from $(n-r)$, one less than $(n-r+1)$, to 1;

$$\text{i.e. by } (n-r)!.$$

(3) It follows that $${}_nP_r = \frac{n!}{(n-r)!}. \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

This is a more convenient form than that given in equation (i).

§**3.** (1) To find the number of combinations of n things taken r at a time (${}_nC_r$).

The number of permutations is $\dfrac{n!}{(n-r)!}$.

Now, as intimated in §**2** (i) (ii) and (iii), the same set of letters will occur in more than one permutation, but in different order.

In fact the number of times the same set of letters will occur is the number of permutations of r things taken r at a time, i.e. by 2 (ii), $r!$

In combinations, these $r!$ arrangements will count as one combination only, and as this is true for all such sets in the permutations of n things, r at a time, we have:

$${}_nC_r = \frac{{}_nP_r}{r!},$$

$$\text{i.e. } {}_nC_r = \frac{n!}{(n-r)!\,r!}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\text{e.g. } {}_nC_3 = \frac{{}_nP_3}{3!} = \frac{{}_nP_3}{3\times 2\times 1} = \frac{{}_nP_3}{6}. \quad \text{(See 1 (iii).)}$$

(2) To show that ${}_nC_r = {}_nC_{n-r}$.

(i) By formula: $${}_nC_r = \frac{n!}{(n-r)!\,r!}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Also, $${}_nC_{n-r} = \frac{n!}{\{n-(n-r)\}!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Since these results, (ii) and (iii), are the same,

$${}_nC_r = {}_nC_{n-r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

(ii) The equality is readily established by the following consideration.

For every combination of r things, taken from the n things, a combination of $(n-r)$ things is left.

$$\therefore\ {}_nC_r = {}_nC_{n-r}.$$

(3) The following example shows the application of combinations to problems on probability or chance.

If 52 cards are dealt to 4 players, what chances are there that a player will receive 4 aces?

13 cards can be dealt in ${}_{52}C_{13}$ ways. Of these, the number of times the 4 aces and 9 other cards are dealt, is ${}_{48}C_9$.

Hence the chances of one player receiving 4 aces are $\dfrac{{}_{48}C_9}{{}_{52}C_{13}}$,

$$\text{i.e. } \frac{48!}{39!\,9!} \div \frac{52!}{39!\,13!} = \frac{11}{4165}, \text{ i.e. 11 in 4165.}$$

(4) The chief importance of combinations is in their application to the expansion of binomials.

Exercise XXV (a)

1. Calculate: ${}_8P_2$, ${}_8P_8$, ${}_5P_3$, ${}_6P_3$, $\dfrac{{}_8P_2}{{}_8P_4}$, $\dfrac{{}_{10}P_6}{{}_6P_2}$.

2. Calculate: ${}_8C_2$, ${}_8C_4$, ${}_5C_3$, ${}_6C_5$, $\dfrac{{}_6C_3}{{}_6C_2}$, $\dfrac{{}_5P_2}{{}_5C_2}$.

3. In how many different orders can 5 counters of different colour be picked up from a table?

4. How many different amounts can be paid out of a till containing a sovereign, half-sovereign, crown, half-crown, florin, shilling, sixpence, three-penny piece, penny, and a half-penny, if three coins only are to be paid out?

5. There are 20 competitors for a first, second or third prize. In how many ways may the prizes be won?

6. How many different whole numbers of three digits can be formed from the figures (i) 1 to 5, (ii) 0 to 5?

7. In how many ways could 5 playing cards be dealt from a pack of 52? How many sets would contain cards of particular different numbers, independent of suit?

8. How many words of three letters can be formed from 6 consonants and 4 vowels, each word to contain one vowel?

9. In how many ways can 6 different wires be connected one to each of 6 terminals?

10.* In how many different orders could 5 men be seated round a circular table? (Place one man, then select the orders of the remaining four. If clockwise and anti-clockwise arrangements are considered alike, halve the result.)

§4. The Binomial Theorem.

(1) To find $(a+b)^n$.

That is, to find the product of n factors, each of which is $(a+b)$.

(i) Clearly, the first term is a^n, and the last b^n.

(ii) The second is of the kind, $a^{n-1}b$.

The term is formed by selecting one b from the n b's available and placing it with the product of the $(n-1)$ a's of the brackets from which the b is not chosen.

Since the b can be chosen in n ways, there are n such products, and the coefficient of the term is therefore n.

The term is thus, $na^{n-1}b$.

(iii) The third term is of the kind $a^{n-2}b^2$.

Two b's can be selected from the n b's in ${}_nC_2$ ways.

(It does not matter in what order they are chosen.)

The third term is, therefore, ${}_nC_2\, a^{n-2}b^2$.

(iv) Similarly, the coefficients of the following terms are ${}_nC_3$, ${}_nC_4$, ${}_nC_5$, etc., the last being ${}_nC_n$, which is, of course, 1, and which agrees with the statement in (i).

(v) Since ${}_nC_1 = n$, we can write the expansion in the form:

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + {}_nC_3a^{n-3}b^3 + \dots$$
$$\dots + {}_nC_ra^{n-r}b^r + \dots + {}_nC_nb^n$$
$$= a^n + na^{n-1}b + \frac{n!}{(n-2)!\,2!}\,a^{n-2}b^2 + \frac{n!}{(n-3)!\,3!}\,a^{n-3}b^3 + \dots$$
$$\dots + \frac{n!}{(n-r)!\,r!}\,a^{n-r}b^r + \dots + b^n$$
$$= a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}\,a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,a^{n-3}b^3 + \dots + b^n.$$

This will be found to agree with the rule given on page 232.

Observe that $\frac{n!}{(n-r)!\,r!}\,a^{n-r}b^r$ is the $(r+1)$th term.

(2) The following are important properties of the terms of the expansion of $(a+b)^n$:

(i) The coefficients of terms equidistant from the beginning and the end are equal.

Thus, ${}_nC_2$ and ${}_nC_{n-2}$ are the coefficients of the terms $a^{n-2}b^2$ and a^2b^{n-2}, which are equidistant from the beginning and the end, and by § **3** (2), page 321,

$$ {}_nC_2 = {}_nC_{n-2}. $$

(ii) The sum of the coefficients is 2^n.

This is found by putting a and b each equal to 1.

(iii) The greatest *coefficient.*

The coefficient of the rth term is ${}_nC_{r-1}$,

,, ,, $(r+1)$th term, ${}_nC_r$,

,, ,, $(r+2)$th term, ${}_nC_{r+1}$.

For the coefficient of the $(r+1)$th term to be the greatest, we must have:

$$ \text{(i)}\ \frac{{}_nC_r}{{}_nC_{r-1}} > 1 \quad \text{and} \quad \text{(ii)}\ \frac{{}_nC_r}{{}_nC_{r+1}} > 1. $$

From (i),

$$ \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r+1)!\,(r-1)!}} > 1, $$

$$ \frac{n-r+1}{r} > 1, $$

$$ n-r+1 > r, $$

$$ n+1 > 2r, $$

$$ \frac{n+1}{2} > r. $$

From (ii),

$$ \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r-1)!\,(r+1)!}} > 1, $$

$$ \frac{r+1}{n-r} > 1, $$

$$ r+1 > n-r, $$

$$ r > \frac{n-1}{2}. $$

That is, for the coefficient of the $(r+1)$th term to be the greatest, r must be greater than $\frac{n-1}{2}$ and less than $\frac{n+1}{2}$; e.g. if n is 8, r must be greater than $3\frac{1}{2}$ and less than $4\frac{1}{2}$.

Hence, r is 4 and the $(r+1)$th term the 5th.

The actual coefficient is ${}_8C_4$, i.e. 70.

EXERCISE.—Verify this by calculating the neighbouring coefficients.

If n is 7, r must be greater than 3 and less than 4; but r must be an integer, and it will be found that both values satisfy the conditions. That is, the 4th and 5th terms have equal coefficients, and the greatest.

(iv) The greatest term of the expansion.

Similar reasoning shows that for the $(r+1)$th term to be the greatest,

(i) $$\frac{{}_nC_r a^{n-r} b^r}{{}_nC_{r-1} a^{n-r+1} b^{r-1}} > 1.$$

From (i), $$\frac{n-r+1}{r} \cdot \frac{b}{a} > 1,$$

$$\frac{b}{a+b}(n+1) > r.$$

(ii) $$\frac{{}_nC_r a^{n-r} b^r}{{}_nC_{r+1} a^{n-r-1} b^{r+1}} > 1.$$

From (ii), $$\frac{r+1}{n-r} \cdot \frac{a}{b} > 1,$$

$$r > \frac{bn-a}{a+b}.$$

EXERCISE.—Verify this, when n is 8, a 2, and b 3. Also, when n is 7, a 2, and b 3.

(3) The expansion of $(a-b)^n$ is obtained by substituting $-b$ for b in the expansion of $(a+b)^n$. It will be found that the two differ in signs only. In the expansion of $(a-b)^n$, the signs run alternately plus and minus.

EXERCISE XXV (B)

1. Find the coefficient of the 7th term in the expansion of $(a+b)^{10}$.
2. Find the coefficient of x^5 in the expansion of $(2-x)^8$.
3. Find the sum of the coefficients in the expansion of $(a+x)^6$.
4. Find the sum of the coefficients in the expanded forms of $(2+x)^5$ and $(2-x)^5$.
5. How many terms are there in the expansion of $(a+b)^9$? Which term has the greatest coefficient, and what is its value?
6. Expand $(1+x)^{10}$, and find the greatest term when $x=2$.

7. Expand $\left(x+\frac{1}{x}\right)^6$ and $\left(x-\frac{1}{x}\right)^6$, and simplify the results.

8. Expand $(a^2+b^2)^5$ and $(\sqrt{a}+\sqrt{b})^6$.

9. Expand $\left(xy-\frac{x}{y}\right)^4$.

10. Expand $(a+b-c)^5$. Arrange it in the form $\{(a+b)-c\}^5$.

11. In the formula for the expansion of $(a+b)^n$, substitute $-n$ for n, and so obtain a formula for the expansion of $(a+b)^{-n}$ or $\frac{1}{(a+b)^n}$.

12. From the formula obtained in Exercise 11, write down a few terms of the expansion of $(1+x)^{-1}$ and of $(1-x)^{-1}$.

Knowing that $(1\pm x)^{-1}=\frac{1}{1\pm x}$, check the terms written down by actually dividing 1 by $1\pm x$.

13. Expand $(a+b)^{-2}$ to four terms, and check your answer as in Exercise 12.

14. Expand to six terms, $(a-b)^{-3}$.

15. In the formula for the expansion of $(1+x)^n$, substitute $\frac{1}{2}$ for n, and so obtain an expression for $(1+x)^{\frac{1}{2}}$ or $\sqrt{1+x}$.

Check your result by finding, say, the first three terms of the square root of $(1+x)$ by the usual method, using fractional indices. (The second term of the root is found by dividing x by 2.)

16. By substituting $\frac{1}{x}$ for n in the formula for the expansion of $(a+b)^n$, expand $(a+b)^{\frac{1}{x}}$.

17. Expand $\left(1+\frac{x}{n}\right)^n$.

18. In Exercise 12, it has been found that

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\text{etc.}$$

The series will be recognised to be a geometric progression, and if x is less than unity the terms decrease in value. What, therefore, is the sum of the series to infinity as calculated by the formula given in Chapter XXI?

19. To compare the following series, namely,

$$\text{(i)}\ x+x^2+x^3+x^4+\text{etc.},$$

$$\text{(ii)}\ x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\text{etc.},$$

when x lies between 0 and 1, take x equal to a fraction, say $\frac{1}{2}$ or $\frac{1}{3}$, and it will be seen that the terms of (ii) are always less than the corresponding terms of (i), and that the terms of (ii) decrease more rapidly than those of (i).

Finally, show that, when x is less than unity, the value of series (ii) is less than $\dfrac{x}{1-x}$.

20. Show that, when x is less than unity, the series,

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\text{etc.},$$

is less than $\dfrac{x}{1-x^2}$.

Then show that, when x is $\frac{1}{3}$, the value of the series is less than ·375.

§ 5. The Exponential Series.

The object is to find a base, and a series in ascending powers of x, such that the base raised to the power x, is equal to the sum of the series.

If z represents the base, the object is to find z, a, b, c, etc., such that

$$z^x = ax^0+bx^1+cx^2+dx^3+\text{etc.} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Since $x^0=1$, equation (i) becomes

$$z^x = a+bx+cx^2+dx^3+\text{etc.} \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

Similarly,

$$z^y = a+by+cy^2+dy^3+\text{etc}, \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

$$z^{(x+y)} = a+b(x+y)+c(x+y)^2+d(x+y)^3+\text{etc.} \quad \ldots\text{(iv)}$$

In equation (iv), $(x+y)$ takes the place of x in equation (ii).

Now, by the rule of multiplication,

$$z^{(x+y)} = z^x \times z^y,$$

and therefore $z^{(x+y)}$ equals also the product of the right sides of equations (ii) and (iii),

i.e. $z^{(x+y)} = (a+bx+cx^2+dx^3+\text{etc.})(a+by+cy^2+dy^3+\text{etc.})$.

Multiplying out,

$$=[a^2+ab(x+y)+\{ac(x^2+y^2)+b^2xy\}$$
$$+\{ad(x^3+y^3)+bc(x^2y+xy^2)\}+\text{etc.}]. \quad \ldots\ldots(\text{v})$$

It follows that the right sides of equations (iv) and (v) are equal, and, comparing the terms, we have:

(1) $a^2=a$. (2) $ab(x+y)=b(x+y)$.

(3) $\{ac(x^2+y^2)+b^2xy\}=c(x+y)^2$.

(4) $\{ad(x^3+y^3)+bc(x^2y+xy^2)\}=d(x+y)^3$, etc.

From (1), $a=1$ or 0; from (2), $a=1$; from (3), since $a=1$, $b^2=2c$, and therefore, $c=\frac{b^2}{2}$; from (4), since $a=1$,

$$bc(x^2y+xy^2)=3d(x^2y+xy^2),$$

and therefore, $d=\frac{bc}{3}=\frac{b^3}{2\times 3}$.

Similarly, the coefficient of x^4 would be found to be

$$\frac{b^4}{2\times 3\times 4}=\frac{b^4}{4!}.$$

Hence, equation (ii) becomes:

$$z^x=1+bx+\frac{b^2x^2}{2!}+\frac{b^3x^3}{3!}+\text{etc.} \quad \ldots\ldots\ldots\ldots\ldots(\text{vi})$$

Now let the value of z, for which b would be equal to unity, be denoted by e. Then

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\text{etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{vii})$$

The value of e can be found by putting x equal to 1, when we have:

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\text{etc.} \quad \ldots\ldots\ldots\ldots\ldots(\text{viii})$$

In Exercise XXI (G), No. 8, it has been found that the sum of an infinite number of terms of this series is less than 3.

By working out the fractions, it will be found that to five places of decimals $e=2{\cdot}71828$.

Hence, equation (vii) satisfies the object, the value of z being $e=2{\cdot}71828$ and the series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\text{etc.}$$

Note.—The exponential series is actually convergent for all finite values of x; that is, the sum of an infinite number of terms tends to a definite value, or has a limit. This value depends, of course, on the value of x. The ratio of the $(n+1)$th term to the nth term is

$$\frac{\frac{x^n}{n!}}{\frac{x^{n-1}}{(n-1)!}}=\frac{x}{n},$$

which can be made less than unity, and, in fact, as small as we please, by taking n large enough. That is, the terms ultimately diminish without limit.

§ **6.** *To calculate Napierian logarithms, that is, logarithms to the base e.*

Take any power n, of z; then, representing z as a power of e, we have:

$$z^n=e^{(n\log_e z)},$$

where $\log_e z$ is the index of the power to which e must be raised to give z. Then, by equation (vii),

$$z^n=e^{(n\log_e z)}=1+n\log_e z+\frac{(n\log_e z)^2}{2!}+\frac{(n\log_e z)^3}{3!}+\text{etc.} \quad \ldots.(\text{i})$$

Now take the case in which $z=1+x$, where x is less than 1; then

$$\left.\begin{aligned}(1+x)^n=1&+n\log_e(1+x)\\&+\frac{\{n\log_e(1+x)\}^2}{2!}+\frac{\{n\log_e(1+x)\}^3}{3!}+\text{etc.}\end{aligned}\right\} \quad (\text{ii})$$

But by the binomial theorem,

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\text{etc.} \quad (\text{iii})$$

Hence, the right sides of equations (ii) and (iii) are equal.

If the right side of equation (iii) is arranged in ascending powers of n, we can compare the terms of the two equal expressions.

Equation (iii) becomes:

$$\left.\begin{aligned}(1+x)^n=1&+n\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\text{etc.}\right)\\&+n^2\left(\frac{x^2}{2}-\frac{x^3}{2}+\frac{11}{24}x^4,\ \text{etc.}\right)+\text{etc.}\end{aligned}\right\}. \quad \ldots\ldots(\text{iv})$$

Comparing terms, the coefficient of n in (iv), namely,

$$\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right),$$

corresponds to the coefficient of n in equation (ii), namely,

$$\log_e(1+x).$$

Equating these coefficients, we have:

$$\log_e(1+x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right). \qquad \text{(v)}$$

Equation (v) *may be tested for known values. Thus, putting* $x = 0$, *we get the value of* $\log_e 1$, *which we know to be* 0.

Also, in (iv), *the earlier terms of the coefficient of* n^2 *are seen to agree with the earlier terms of half the square of the coefficient of* n, *as suggested by* (ii).

Substituting $-x$ for x in equation (v),

$$\log_e(1-x) = \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{etc.}\right). \qquad \text{(vi)}$$

Again, $$\log_e \frac{1+x}{1-x} = \log(1+x) - \log(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{etc.}\right); \qquad \text{(vii)}$$

and if we put $\frac{M}{N}$ for $\frac{1+x}{1-x}$, then, since $x = \frac{M-N}{M+N}$,

$$\log_e \frac{M}{N} = 2\left(\frac{M-N}{M+N} + \frac{1}{3}\left(\frac{M-N}{M+N}\right)^3 + \frac{1}{5}\left(\frac{M-N}{M+N}\right)^5 + \text{etc.}\right). \qquad \text{(viii)}$$

From equation (viii), logarithms of successive numbers can be calculated to any desired decimal place.

E.g. for $\log_e 2$, put $M = 2$ and $N = 1$; then

$$\log_e \frac{2}{1} = 2\left\{\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \text{etc.}\right\},$$

which determines $\log_e 2$.

For $\log_e 3$, put $M = 3$ and $N = 2$; then

$$\log_e \frac{3}{2} = 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \text{etc.}\right\}.$$

Since $\log_e \frac{3}{2} = \log_e 3 - \log_e 2$ and $\log_e 2$ is known, $\log_e 3$ can be determined.

Complete the calculation.

Exercise XXV (c)

1. By formula (viii), p. 330, calculate correct to the fourth decimal place $\log_e 2$. Then find $\log_{10} 2$, given that

$$\log_e 10 = 2{\cdot}3026.$$

Examine the following:

Let $$x = \log_{10} 2;$$

then $$10^x = 2,$$

Take logs to base e; then $x \log_e 10 = \log_e 2$, from which x is readily determined.

2. Show that $\log_e x = \dfrac{\log_{10} x}{\log_{10} e}$.

3. Find $\log_e 5$ from $\log_{10} 5$.

4. From the series form of e^x, find in similar form e^{-x} and $\dfrac{e^x + e^{-x}}{2}$, and also $\dfrac{e^x - e^{-x}}{2}$.

5. Plot e^{-x^2} for a few positive and negative values of x, and observe the form of the graph.

6. Represent as a series, $\log_e \dfrac{1+x}{1-x}$.

7. Find $e^x \sin x$ when $x = \dfrac{\pi}{4}$.

8. A telephone current diminishes in the ratio e^{-cx} in a distance x miles. Find the current at a point 100 miles from the transmitting station, if at this station the current is 1 unit, and if c is 0·0125.

9. When a rope is wrapped round a round post and a pound weight attached to one end, it is found that the pull P, to be applied at the other end to cause the rope to slip, is given by the equation

$$P = e^{\mu\theta} \text{ (lbs.)},$$

in which μ is the coefficient of friction between the rope and the post, and θ is the angle of lap round the post in radians.

Find P when the rope is lapped twice round and the coefficient μ is 0·25.

10. The velocity v of a projectile at a distance x from the point of projection is given by the formula

$$v = V \cos \alpha \, . \, e^{-\frac{gx}{\mu}},$$

in which V is the initial velocity, α is the angle of projection, and μ is the acceleration at unit distance.

For what values of α and μ would v equal V?

Find μ when $\frac{v}{V} = \frac{3}{4}$, $x = 1000$, $\alpha = 30°$ and $g = -32$.

11. The equation of the common catenary is

$$y = c \frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2}.$$

Show that near the vertex the catenary approximately coincides with a parabola.

12. If $x = \frac{E}{R} + Ce^{-\frac{Rt}{L}}$, and $x = 0$ when $t = 0$, find C. Then find x when $E = 100$, $R = 20$, $L = 0{\cdot}1$ and $t = 0{\cdot}05$.

Revision Exercise IV

1. Employ the relation $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ to find $\frac{dy}{dx}$, when

(i) $y = (a - x)^n$, (ii) $y = \sin^2 x$, (iii) $y = \cos^2 x$, (iv) $y = \tan^2 x$.

Verify your results by direct differentiation.

2. Differentiate with respect to x, $\sin^n x$, $\cos^n x$, $\tan^n x$.

3. Find the area bounded by the graph $y = \sin^2 \theta$, from $\theta = 45°$ to $\theta = 90°$.

4. A radius, the length of which varies according to the law $r = a + b \sin \theta$, rotates about one end.

Find the area enclosed by the curve described by the other end when the radius makes one complete revolution.

5. The equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

(i) Show that if $\theta = \sin^{-1}\frac{x}{a}$, this equation takes the form $y = b \cos \theta$.

(ii) The area bounded by the curve between the ordinates $x = a$ and $x = 0$ is

$$\int_0^a y \, dx = \int_0^{\frac{\pi}{2}} y \frac{dx}{d\theta} \, . \, d\theta.$$

Show that this integral is equal to

$$\tfrac{1}{2}ab\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta=\frac{\pi}{4}ab,$$

and that the area of an ellipse is therefore πab, where a and b are its semi-axes.

6. Solve Ex. XVIII (B), 26, (3) and (4), by integration.

If l is the length of the rope, and r the radius of the post, then when the rope has turned through an angle θ radians, the unlapped portion is $(l-r\theta)$. The arc described by the free end is the integral of $(l-r\theta)\,d\theta$, and the area swept over by the rope, the integral of $\frac{1}{2}(l-r\theta)^2\,d\theta$.

7. Ex. XVIII (B), 27, is more difficult when the length of the rope, instead of the position of the meeting point, is given.

If θ (radians) is the "angle of lap" of the rope on the post, the length of the lapped rope is $r\theta$, and of the unlapped rope, $r\tan(\pi-\theta)$.

If the total length of rope is 5 ft., and r is 1 ft., then

$$\tan(\pi-\theta)+\theta=5, \quad \text{or} \quad \tan(\pi-\theta)=5-\theta.$$

The value of θ which satisfies this equation can be found from the graphs of $\tan(\pi-\theta)$ and $(5-\theta)$ plotted against θ.

Find this value, and complete the exercise.

8. Employ the exponential series to show that

$$\frac{1}{e}=\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\text{etc.}$$

ANSWERS

Ex. I (A) Page 9

1. + £250 as contrasted with − £250. **2.** − £60. **3.** £60.
4. £115. **5.** − £115. **6.** Assets, £850; debts, £500; state, £350.
7. Zero. **8.** − £150.

Ex. I (B) Page 11

1. (*a*) 12°; (*b*) − 5°; (*c*) 0°; (*d*) 100°; (*e*) − 15°. **2.** 25°. **3.** 10°.
4. − 10°. **5.** 15°. **6.** 10. **7.** 20, − 5, 0, 3, 5. **8.** 180. **9.** 190.

Ex. I (C) Page 12

2. + 2·3 cms. **3.** − 7·9 cms. **4.** − 2·6 cms.

Ex. I (D) Page 13

1. 60°. **2.** + 20°. **4.** + 3028 ft., − 900 ft.
5. Clockwise, + 60 r.p.m., − 60 r.p.m. **6.** − 100 r.p.m., − 200 r.p.m.
7. + 15 mins., − 15 mins. **8.** 8 mins. per day, − 5 mins. per day.
9. + 32·2 ft. per sec. each sec., − 32·2 ft. per sec. each sec.
10. + 12 lbs., − 12 lbs. **11.** By using plus and minus signs.
12. − 55, + 1915.

Ex. II (A) Page 16

1. 21. **2.** − 11. **3.** − 5. **4.** 5. **5.** 7.
6. 0. **7.** 13, − 5. **8.** − 13. **9.** 3. **10.** − 3.

Ex. II (B) Page 17

1. 20. **2.** − 20. **3.** 4. **4.** − 4. **5.** − 25. **6.** − 15.
7. 15. **8.** 25. **10.** 16. **11.** − 19. **12.** $12\frac{1}{4}$ miles W.

Ex. II (C) Page 20

1. 4. **2.** − 4. **3.** 16. **4.** − 16. **5.** 4. **6.** − 4. **7.** 16. **8.** − 16.
9–12. Regarded from the second number, − 28, − 8, 8, 28. **13.** − 5.
14. 0. **15.** − 4. **16.** − 10. **17.** 10. **18.** 10. **19.** 13. **20.** 13. **21.** 3.
22. 5. **23.** − 5. **24.** 14. **25.** 1. **26.** 9. **27.** 20. **28.** − 4.

Ex. III (A) Page 23

1. 18. **2.** -18. **3.** -18. **4.** 18. **5.** 1. **6.** -1. **7.** -1.
8. 1. **9.** $-\frac{3}{8}$. **10.** 1. **11.** 0. **12.** 0. **13.** 0. **14.** -16.
15. -90. **16.** 60. **17.** -36. **18.** 4. **19.** -8. **20.** -810.
21. $+5\times+3=+15$, $+5\times-3=-15$, $-5\times+3=-15$, $-5\times-3=+15$.
22. (i) minus, (ii) plus.

Ex. III (B) Page 24

1. 5. **2.** -5. **3.** -5. **4.** 5. **5.** $\frac{1}{5}$. **6.** $-\frac{1}{5}$.
7. $-\frac{1}{5}$. **8.** $\frac{1}{5}$. **9.** -6. **10.** 6. **11.** 3. **12.** -3.
13. -2. **14.** 0. **15.** 5. **16.** 5. **17.** 0. **18.** -5.
19. -7. **20.** 4. **21.** -4. **22.** 12. **24.** (i) 6, (ii) $-1\frac{1}{5}$.

Ex. IV (A) Page 28

1. $7a$. **2.** $8x$. **3.** $3x$. **4.** $9a$. **5.** $8y$. **6.** $2a+9c$.
7. $7x$. **8.** $-4x+5$. **9.** $8p$. **10.** $2p$. **11.** $-8p$. **12.** $a+3b$.
13. $3a+2b$. **14.** $-5x+2y$. **15.** $5x-2y$. **16.** $3x$. **17.** $(w+x)$ grms.
18. $(x+100)$ grms. **20.** $8a+11b+6c$. **21.** $-b+4c$. **22.** $2x+a+5b$, 18.
23. $4a-3c=-5$; lines, $+14$, -2, -10, 0, -7. **24.** $8a+7x-8y$.
25. (i) 0, (ii) 12, (iii) -5, (iv) -13, (v) 1, (vi) 5, (vii) $3a+13$, (viii) $c-1\frac{7}{12}$.
26. (i) 7, (ii) -3. **27.** (i) 0, (ii) $6a$, $6b$, or $6c$, (iii) $1\frac{3}{5}a$, $1\frac{3}{5}b$, or $1\frac{3}{5}c$.

Ex. IV (B) Page 30

1. (i) $4a$, (ii) $-4a$, (iii) $8a$, (iv) $-8a$, (v) $-4a$, (vi) $4a$, (vii) $8a$, (viii) $-8a$.
3. $a-7b+8c$. **4.** (i) $2x-2$, (ii) x, (iii) $-x$, (iv) $2x$.
5. $-a-7b-2$, -17. **6.** $-2a+7b-4c$, 29.
7. Regarded from the first term, (i) $2x$, (ii) a, (iii) $3x-3$, (iv) $x+y$. Regarded from the second term, (i) $-2x$, (ii) $-a$, (iii) $3-3x$, (iv) $-x-y$.
8. $(y-x)$, $(z-y)$ grms. **9.** $(w+m+n-x)$ grms., $(w+m+n-x)$ grms.
10. (C $-$ W), (C $-$ M), (M $-$ W) grms.

Ex. IV (C) Page 33

1. ax. **2.** ax. **3.** abx. **4.** $-6ax$. **5.** $-2xy$.
7. $ax+2a$. **8.** $-3x-6$. **9.** $3ax+6a$, $-3ax-6a$.
10. $-6ax+4bx$. **12.** a^2, $-a^2$, $-a^2$, a^2.
14. (i) $-6x^3y^3$, (ii) $3ab^3$, (iii) $4x^2$, (iv) $-4x^2$, (v) $4x^2$, (vi) x^6, (vii) $9a^4b^2$, (viii) $-a^3$, (ix) $8a^6$. (i) -1296, (ii) 9, (iii) 16, (iv) -16, (v) 16, (vi) 64, (vii) 729, (viii) 27, (ix) 5832.
15. $\frac{1}{2}xy$, xy, xy, $\frac{9}{2}bh$, $3b$.

Ex. IV (D) Page 35

1. $-a$. **2.** a. **3.** $3x^3$. **4.** $-\frac{1}{3x^4}$. **5.** $-3a^2$. **6.** $\frac{3}{2}b^3$.

7. $3a^2b$. **8.** $-4xy$. **9.** $a+4b$. **10.** $x-y^2$. **11.** $10x$.

12. $-\frac{1}{2}a^2$. **13.** $\frac{x+y^2}{2}$ or $\frac{1}{2}x+\frac{1}{2}y^2$. **14.** $x-3xy^2$. **15.** $5a^2b^2+b-4$.

16. $-4y^2$. **17.** x^2+5x-4. **18.** $\frac{6}{y}$. **19.** $\frac{5}{y}$. **20.** $\frac{6}{x}$.

21. (i) $-1\frac{1}{3}$, (ii) $-\frac{2}{3}$, (iii) -54, (iv) $1\frac{7}{36}$, (v) 6, (vi) 6.

22. $\frac{1}{x^3}$. **23.** $\frac{a-b}{a+b}$. **24.** 228.

Ex. IV (E) Page 37

1. 1, 1, $4x^2$, $9x^2$, $25x^4$, $9x^6$, $16x$, $-x$, x^3, $9(a+b)^2$.

2. $\pm 5a$, $\pm 3b^2$, $\pm 7b^3$, $\pm 8y^5$, $\pm 5(a+b)$. **4.** $\pm\frac{x}{2y^2}$, $\pm\frac{x}{4y^4}$, $\pm\frac{1}{x}$, $\pm\frac{1}{x}$, $\pm 3x$.

5. ± 4, ± 5, ± 8, ± 10, ± 12, $2\sqrt{3}$, $2\sqrt{5}$, $2\sqrt{7}$, $6\sqrt{2}$, $4\sqrt{2}$, $-5\sqrt{3}$, ± 12, $12\sqrt{2}$, $2x\sqrt{2}$, $2x\sqrt{2}$, $4x^2y\sqrt{2}$, $x\sqrt{2}$.

7. 10^3, 3; 10^2, 2; 10^0, 0; 10^4, 4; 10^6, 6; 10^7, 7.

8. 2, 5, 6, 3, 3, 3. **9.** 2^5, 2^5, 3^5, 5^{10}. **10.** 5.

Ex. IV (F) Page 38

1. (i) $a+b+c+d$, (ii) $4a$, (iii) $2a+2b$, (iv) $2a+2b$, (v) $4a$, (vi) $a+b+c$, (vii) $2a+b$, (viii) $3a$.

2. πd, $2\pi r$. **3.** $\pi r+2r$, $\frac{\pi r}{2}+2r$. **5.** $\frac{ah}{2}+\frac{bh}{2}$. **6.** πr^2, $\frac{\pi r^2}{4}$.

7. $r^2-\frac{\pi r^2}{4}$, $\frac{\pi r^2}{4}-\frac{r^2}{2}$, $\frac{\pi r^2}{2}-r^2$. **8.** lbh, $\frac{abh}{2}$, πr^2h.

9. $\frac{1}{3}abh$, $\frac{1}{6}abh$, $\frac{1}{3}\pi r^2h$. **10.** $\frac{z-y}{y-x}$. **11.** $\frac{y}{x}$ grms., $\frac{px}{y}$.

12. $\frac{n}{s}$, $\frac{nt}{s}$. **13.** $\left(90-\frac{x}{2}\right)^\circ$. **14.** $\frac{l-a-b-c}{2}$. **15.** $\frac{c}{10}$ mms.

Ex. V (A) Page 49

5. 150 sovs. and £40 debts. **6.** $10x-15y$. **7.** $-10x+15y$.

8. $2x^2-3xy$. **9.** $-2x^2+3xy$. **10.** $-2xy-3y^2$. **11.** $-2xy+3y^2$.

12. $a^2+b^2+c^2$. **13.** $2a-8b+9c$. **14.** $3x^2-2xy-3x-14y$. **15.** $12a-3b$.

16. $-4a+6b$. **17.** 0. **18.** $2(x+y)$. **19.** $-2(x+y)$.

20. $-2(x-3y)$. **21.** $3(a-2b+4c)$. **22.** $3(a-2b)+5(c+5d)$.

23. $3(a-2b)-5(c-5d)$. **24.** $a(a-b)-c(c-d)$. **25.** $a(a+b)-b(a-b)$.

26. $x(x^2-3xy-y^2)$. **28.** $x-2y$. **29.** $3x^2+4y^2$. **30.** $2x-5y$.

Ex. V (B) Page 51

1. $-13x-3y$. **2.** $-a-ax+5ay+4b$. **3.** $2a-3$. **4.** $-34x+48$.
5. 27. **6.** $13x-y-4$. **7.** $2a\{3a-b(a-b)\}$. **8.** $a\{a(x^2-y^2)+xy\}$.
9. $a\{b-c(b-1)\}$. **10.** $x(ax+b)-y(cy-d)$. **11.** $a(x+y+c)$.
12. $(a+b)(a-b)$. **13.** $(p+q)(x+y+c)$. **14.** $(p+q)(x+2y+z)$.
15. $2y(p+q)$. **16.** $(a+b)(a+b)$. **17.** $2(a+b)$, 12·2 ins.
18. $2\{h(a+b)+ab\}$. **19.** 205·06. **20.** 3·8104.

Ex. V (C) Page 54

1. x^2y, x^3y^2. **2.** ab^2c^3, $a^2b^3c^4$. **3.** a^3, a^4b^3. **4.** $5x^3y^3$, $30x^4y^4$.
5. $3abc^2$, $6a^2b^2c^3$. **6.** $a^3b^3c^3$, $a^5b^5c^5$. **7.** 1, $6abc$. **8.** 1, $6ac$.
9. 5, 50 abc. **10.** $a+b$. **11.** $5+3d$. **12.** $\frac{3}{2}$.
13. $\frac{a+2}{x}$. **14.** $\frac{2a+b}{2x}$. **15.** $\frac{a^2+x^2}{ax}$. **16.** $-\frac{1}{2}$. **17.** $\frac{3y}{2(a-b)}$.
18. $\frac{a}{c}$. **19.** $-\frac{b}{c}$. **20.** $\frac{x^2+2y^2}{3xy}$. **21.** $\frac{3x^2+4y^2}{6xy}$. **22.** $-\frac{5a+7}{12}$.
23. $\frac{9y-7x-20}{10}$. **24.** $\frac{2a^2}{3b}$. **25.** $\frac{3x}{ay^2}$. **26.** 45.
27. $\frac{b(ax+1)}{a(bx+1)}$. **28.** $\frac{a}{b}$. **29.** $ab(a+b)$. **30.** $6ab(a+1)$.

Ex. VI (A). Page 58

1. 4. **2.** 9. **3.** -3. **4.** 0. **5.** 0. **6.** -10.
7. 5. **8.** -5. **9.** -5. **10.** -3. **11.** -3. **12.** 3.
13. $3\frac{1}{3}$. **14.** $-3\frac{1}{3}$. **15.** $-3\frac{1}{3}$. **16.** $3\frac{1}{3}$. **17.** $\frac{1}{8}$. **18.** $-\frac{1}{8}$.
19. $\frac{1}{8}$. **20.** $-\frac{1}{8}$. **21.** -3. **22.** -3. **23.** -3. **24.** 5.
25. 5. **26.** 1. **27.** 3.

Ex. VI (B) Page 60

1. 6. **2.** 3. **3.** -3. **4.** 3. **5.** $\frac{1}{2}$. **6.** 4.
7. -4. **8.** -4. **9.** 12. **10.** 3. **11.** -3. **12.** $\frac{1}{3}$.
13. -1. **14.** $\frac{3}{2}$. **15.** 1. **16.** $2\frac{7}{9}$. **17.** $\frac{9}{25}$. **18.** -3.
19. 10. **20.** -1. **21.** $20\frac{1}{5}$. **22.** 21. **23.** 5. **24.** 60. **25.** 135.
26. $20\frac{1}{2}$. **27.** $26\frac{2}{3}$. **28.** -3. **29.** 2. **30.** 3. **31.** £11. 17s. 6d.
32. (i) £3·176 + ·01d., (ii) £3·377 + ·02d., (iii) £3·877 + ·02d. (iv) £3·384 + ·09d.
(i) £635. 4s. 2d., (ii) £755. 8s. 4d., (iii) £775. 8s. 4d., (iv) £676. 17s. 6d.
33. 11 a.m., 5 a.m., 7 a.m., 5 p.m., 11.20 p.m.

Ex. VII (A) Page 62

1. $ac - ad + bc - bd.$ **2.** $ac + ad - bc - bd.$ **3.** $6ac + 4ad + 9bc + 6bd.$

4. $6ac - 4ad + 9bc - 6bd.$ **5.** $6ac - 4ad - 9bc + 6bd.$

6. $6m^2p^3 - 9m^2q^3 - 10n^2p^3 + 15n^2q^3.$ **7.** $x^2 - 5x + 6.$ **8.** $x^2 + 5x + 6.$

9. $x^2 - x - 6.$ **10.** $x^2 + x - 6.$ **11.** $4x^2 - 4xy - 3y^2.$ **12.** $12x^2 - 26x + 12$

13. $ax - a^2.$ **14.** $x^3 - a^2x.$ **15.** $ac + ad + ae + bc + bd + be.$

16. $a^2 - b^2 + ac + bc.$ **17.** $ab - 4a + 3b - 12.$ **18.** $10x^2y^2 - 19xy - 15.$

19. $x^3 + x^2 - 7x + 2.$ **20.** $2a^4 - 7a^3 + 10a^2 - 7a + 2.$

21. $x^4 + x^3y - 13x^2y^2 - xy^3 + 12y^4.$ **22.** $x^3 - \frac{1}{x^3}.$

23. $a^3 - b^3 + c^3 + 3abc.$ **24.** $4 + 4x - 9x^2 + 5x^3 - 15x^4 + 14x^5 - 6x^6.$

Ex. VII (B) Page 64

1. $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, $a^2 - b^2$. **2.** $4a^2 + 4ab + b^2$, $4a^2 - 4ab + b^2$, $4a^2 - b^2$.

3. $a^2 + 4ab + 4b^2$, $a^2 - 4ab + 4b^2$, $a^2 - 4b^2$.

4. $4x^2 + 20x + 25$, $4x^2 - 20x + 25$, $4x^2 - 25$. **5.** $\frac{x^2}{4} + x + 1$, $\frac{x^2}{4} - x + 1$, $\frac{x^2}{4} - 1$.

6. $4x^2 + 12xy + 9y^2$, $4x^2 - 12xy + 9y^2$, $4x^2 - 9y^2$. **7.** 289, 2209, 2475.

8. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$, $a^2 + b^2 + c^2 + 2ab - 2bc - 2ac$, $a^2 + 2ab + b^2 - c^2$.

9. $4a^2 + b^2 + 4c^2 - 4ab - 4bc + 8ac$, $4a^2 + b^2 + 4c^2 - 4ab + 4bc - 8ac$, $4a^2 + b^2 - 4c^2 - 4ab$.

10. $144a^2 + 144ab + 36b^2$, $a^4 - 2a^2b^2 + b^4$.

Ex. VII (C) Page 66

2. $a + b.$ **3.** $a - b.$ **4.** $a - b.$ **5.** $a + b.$ **6.** $a - b$, rem. $2b^2$.

7. $a + b$, rem. $2b^2$. **8.** $x - 2.$ **9.** $a^2 - 2a + 2.$ **10.** $3x^2y^2 + xy - 4.$

11. $x^2 + x + 3.$ **12.** $a^3 - 2a^2b + 4ab^2 - 8b^3.$ **13.** $3 + a - a^2.$

14. $2x^2 + xy - 3y^2.$ **15.** $x^2 + 4x + 8.$ **16.** $1 + 2a + a^2.$

17. $2x - 2$, $x = 1$. **18.** 6.

Ex. VII (D) Page 67

1. (i) b^2, (ii) b^2, (iii) $\pm 2ab$. **2.** (i) $4y^2$, (ii) $4y^2$, (iii) $\pm 4xy$.

3. (i) a^2, (ii) a^2, (iii) 1. **4.** (i) x^2, (ii) y^2, (iii) 1.

5. (i) $4y^2$, (ii) $4y^2$, (iii) 4. **6.** (i) $\frac{1}{4}$, (ii) $\frac{1}{4}$, (iii) ·16. **7.** $25y^4$.

8. 9. **9.** $9b^2y^2$. **10.** y^2. **11.** b^2. **12.** $\frac{4}{y^2}$.

Ex. VII (E) Page 70

1. $2a+5b$. **2.** $2a-5b$. **3.** $4x-5y$. **4.** $6x^2+1$. **5.** a^2-2a-2.
6. $1-2y+y^2$. **7.** $x^2-xy+2y^2$. **8.** $3a^2-2a+1$. **9.** 99. **10.** 123.
11. 55·5. **12.** 1·414. **13.** 1·732. **14.** 2·236. **15.** 2·449.
16. 2·646. **17.** 2·874. **18.** ·196. **19.** 15·215. **20.** 39·795.
21. (i) 13 cms., (ii) 25 cms., (iii) 41 cms., (iv) 85 ins., (v) 65 ft., (vi) 37 cms., (vii) 4·3 ins.
22. (i) 12 cms., (ii) 55 cms., (iii) 108 cms., (iv) 20 ins., (v) 60 ins., (vi) 77 ft., (vii) 3 cms., (viii) 27·7 ft.
23. 19·4 ft. **24.** 11·31 cms. **25.** 18 cms.

Ex. VIII (A) Page 72

1. $\frac{5}{6}, \frac{9}{8}, \frac{5}{3}, \frac{3}{4}, \frac{65}{44}, \frac{3}{5}, \frac{2}{3}, \frac{5}{6}$. **3.** 4 : 1. **4.** 4 : 1. **5.** 1 : 12, no.
6. 4 : 5. **7.** 22 : 7, 11 : 14 (approx.). **8.** $\frac{4-\pi}{16}=\frac{3}{56}$ (approx.).
9. $\frac{5}{8}, \frac{5}{9}, \frac{6}{11}, \frac{1}{2}, \frac{3}{7}, \frac{2}{5}$. **10.** $\frac{M-B}{W-B}$.

Ex. VIII (B) Page 75

1. $8\frac{1}{3}$. **2.** $2\frac{2}{5}$. **3.** $8\frac{3}{4}$. **4.** $22\frac{2}{5}$. **5.** $\frac{ab}{c}$. **6.** $\frac{ac}{b}$. **7.** $\frac{a^2}{b}$. **8.** b.
9. (i) directly, (ii) inversely, (iii) directly, (iv) inversely.
10. (i) 3, (ii) $8\frac{1}{3}$, (iii) 12, (iv) $22\frac{1}{2}$. **11.** (i) inversely, (ii) directly.
12. 32,800 sq. mls. **13.** $\frac{Rx}{180}$. **14.** 2·618 cms. **16.** $\frac{1}{6}$.
17. $\frac{x}{360}, \frac{\pi R^2 x}{360}$. **18.** $\frac{\pi R^2}{12}, \frac{\pi R^2}{8}, \frac{\pi R^2}{6}$, etc.
19. Ratios inversely equal. **20.** Ratios inversely equal, $37\frac{7}{9}^\circ$.
21. Ratios inversely equal. **22.** 128·3 grms.
23. (i) acid : alkali $= y : x$, (ii) 1st acid : 2nd acid $= z : x$. **24.** 7·854 ins.
25. 57·3°, 28·65°. **26.** $\frac{11x}{12}$. **27.** $\frac{A}{A-W}$. **28.** $\frac{A-T}{A-W}$.
29. $(W+C-G)$ grms. **32.** 4, inversely. **34.** 1st : 2nd = 3 : 1.

Ex. VIII (C) Page 84

6. 56′ 3″. **7.** 2″. **8.** 13 cms.

Ex. IX (B) Page 87

1. 5·196, 6 ins.
2. 3·81, 5 cms.
3. 1·5, 2·598 ins.
4. 2·06, 2·45 cms.
5. 45·9, 45·72 ins.
6. 1·607, 1·915 ins.
7. 7·8 sq. ins., 6·11 sq. cms., 1·95 sq. ins., 2·52 sq. cms., 91·4 sq. ins., 1·53 sq. ins.
8. 251·7 ft.
9. 37° (approx.).
10. 122 yds.

Ex. IX (C) Page 90

3. 1·48, 2·27 cms.
4. 1·76, 1·31 cms.
5. 2·69, 3·06 cms.
6. 7·66, 6·43 cms.
7. 7·72, 3·92 ins.
8. 815·1, 921·5 yds.
10. 1·965, 1·13, 2·65, 24·6 sq. cms.; 11·6 sq. ins.
11. 15,466 miles, 644·4 miles per hour.
13. 120 yds.
14. 176·9 yds.
15. $7\frac{1}{2}$ million sq. miles.
16. $1\frac{1}{2}$ million sq. miles.
17. 3750 miles.
18. 4428 miles.
19. 84°, 4·97 ft.; 192°, 168°, 17·9 ft.; 150 r.p.m., 15·7.
20. 21·29 ins.

Revision Ex. I Page 92

1. (i) -4, (ii) 12, (iii) $7(a+b)$, (iv) $4a(a-b)-2(x+y)$.
2. (i) -20, (ii) 6, (iii) $-8x^2+8x+3$, (iv) $(a-3b)$.
3. 0·9 in.
4. (i) 0, (ii) 1, $\frac{1}{4}$.
5. (i) $s\left(\frac{\pi}{2}+2\right)$, (ii) $\frac{s^2}{8}(\pi+2\sqrt{3})$.
6. 94.
7. $a^2+4x^2+9y^2+4ax-12xy-6ay$.
8. (i) 21, (ii) -27, no.
9. (i) -6, (ii) 9.
11. 6·34, 12·68, 10·98 ins., 34·8066 sq. ins.
12. (i) 0·183, $\frac{1}{2}$, (ii) 1·618.

Ex. X (A) Page 100

1. (i) -5, $1\frac{2}{3}$; (ii) 4, 4; (iii) -7, $10\frac{1}{2}$.
2. (i) $y=3x-7$, (ii) $y=-3x+5$, (iii) $y=5x-3$, (iv) $y=-2x+12$, (v) $y=\frac{5}{2}x$.
3. $y=5$.
4. $y=\frac{1}{2}x+5$, $y=\frac{1}{2}x-5$, $y=-\frac{1}{2}x+5$, $y=-\frac{1}{2}x-5$, $y=5$, $y=-5$, $x=5$, $x=-5$.
7. The graph becomes steeper, and finally is vertical.
8. $y=-0{\cdot}1x-3$, $y=0{\cdot}4x+30{\cdot}1$.

Ex. X (B) Page 102

3. $2\frac{3}{4}$, $-\frac{1}{2}$.
10. 14th.
11. 8th.
12. (i) 4·48 p.m., (ii) 4·50 p.m. Thursday, $y=10x-30$, $y=50-15x$.
13. $y=2x+3$.
14. $y=\frac{3}{2}x-6$.
15. $y=-2x+3$.
16. $y=\frac{1}{2}x$.
17. 5.
18. $y=-x+3$.

Ex. X (c) Page 106

1. $f=0{\cdot}8h+8$. **3.** $f=0{\cdot}217w+1{\cdot}5$. **4.** $R=0{\cdot}073D+30$.
5. Temp. $=3t+15$. **6.** $L=-0{\cdot}687t+605{\cdot}7$.
7. $R=0{\cdot}344t+100$. **8.** $V=0{\cdot}0128t+1$.

Ex. X (D) Page 112

4. $V=\frac{450}{P}$. **5.** $D=\frac{500}{W}$. **6.** $y=\frac{3}{x}+2$, $x=0$, $y=$

Ex. XI (A) Page 117

1. $11\frac{1}{5}$, $7\frac{4}{5}$. **2.** 3, 2. **3.** 6, 4. **4.** $\frac{5}{6}$, $-\frac{5}{6}$.
5. $1\frac{8}{19}$, $1\frac{7}{19}$. **6.** 5, 7. **7.** $\frac{1}{10}$, 10. **8.** 2, $\frac{1}{3}$.
9. 3, 2. **10.** $8\frac{3}{7}$, $3\frac{2}{7}$, 3. **11.** $\frac{5}{2}$, $\frac{1}{2}$. **12.** 20, 20.
13. 17·4, 12. **14.** −2, 1. **15.** 9, 11, 13.
16. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$. **17.** 4, 0, 5. **18.** $7\frac{1}{2}$, −4.

Ex. XI (B) Page 118

1. $a+b+1$. **2.** $\frac{3a^3b^2}{5c}$. **3.** $b-c$. **4.** a, b.
5. 1. **6.** a, b. **8.** $\frac{c}{2\pi}$. **9.** $2\sqrt{\frac{A}{\pi}}$.
10. $\frac{c}{\pi}-b$. **11.** $\frac{A}{\pi a}$. **12.** $\frac{1}{2}\sqrt{\frac{A}{\pi}}$. **13.** $\sqrt[3]{\frac{3V}{4\pi}}$.

14. $s=\frac{Q}{W(t_1-t_2)}$, $t_1=\frac{Q}{Ws}+t_2$, $t_2=t_1-\frac{Q}{Ws}$.

15. $x=\frac{WsT+wt}{Ws+w}$, $w=\frac{Ws(T-x)}{x-t}$, $t=\frac{x(Ws+w)-WsT}{w}$, $s=\frac{w(x-t)}{W(T-x)}$.

16. $L=\frac{w(x-t)}{W}-(T-x)$, $x=\frac{W(L+T)+wt}{W+w}$.

17. $L=\frac{(w_1+w_2s)(x-t)}{W}-(T-x)$, $s=\frac{W(L+T-x)}{w_2(x-t)}-\frac{w_1}{w_2}$,
$x=\frac{(w_1+w_2s)t+W(L+T)}{w_1+w_2s+W}$.

18. $a=\frac{L-l}{lt}$. **19.** $D=\frac{d}{1+bt}$. **20.** $k=\frac{Hl}{A(T-t)}$, $T=\frac{Hl}{kA}+$

21. $a=\frac{\sqrt{2(s-ut)}}{t}$, $u=\frac{s}{t}-\frac{1}{2}at$, $s=16t^2$.

22. $s=\dfrac{v^2-u^2}{2a}$, $a=\dfrac{v^2-u^2}{2s}$, $v=\sqrt{u^2+2as}$.

23. $v=\dfrac{Ft}{m}+u$.

24. $F=\dfrac{W}{2s}(v^2-u^2)$, $g=\dfrac{v^2-u^2}{2s}$.

26. $m=\dfrac{FR^2r^2}{R^2-r^2}$.

27. $f=\dfrac{uv}{u-v}$, $v=\dfrac{uf}{u+f}$.

28. + when $u<f$, − when $f<u$.

29. $A=1$, $B=2$.

30. $a=-124\frac{2}{3}$, $b=\frac{31}{30}$, 94·4.

31. $\dfrac{lt(x+y)}{tx+ly}$.

Ex. XI (C) Page 122

1. $\dfrac{88x}{3y}$.	**2.** $\dfrac{88x-3y}{60}$.	**3.** 666·4x.	**4.** 5·25x.
5. 9.	**6.** $4\frac{8}{9}$, $7\frac{1}{9}$ ins.	**7.** 2, 24.	**8.** 30 m.p.h.
9. 300 r.p.m.	**10.** 9, 11, 13, 15.	**11.** 3, 9.	**12.** 32, 21.
13. 11·4, 7·3.	**14.** 20 × 15 yds.	**15.** 3.	**16.** $57\frac{1}{2}$.

Ex. XII (A) Page 124

1. $x(x-1)$.
2. $a(1-x)$.
3. $a(a+x)$.
4. $ax(a-x)$.
5. $x(x^2+1)$.
6. $(2a+3c)(a-2b)$.
7. $(a-3)(x-y)$.
8. $(a^3+3)(a-1)$.
9. $(2y-5)(x^2+a)$.
10. $(x+3)(x-a)$.
11. $(a^3-2b^3)(a+b)$.
12. $2x(x^2-4)(x-3)$.
13. $x(a^4+4bc-b^2-4c^2)$.
14. (i) $(a-b)(a^2+ab+b^2)$, (ii) $(a+b)(a^2-ab+b^2)$.

Ex. XII (B) Page 126

1. $(x-3)(x-2)$.
2. $(x+3)(x+2)$.
3. $(x+3)(x-2)$.
4. $(x+6)(x+1)$.
5. $(x-3)(x+2)$.
6. $(x-6)(x-1)$.
7. $(x+6)(x-1)$.
8. $(x-6)(x+1)$.
9. $(a+12)(a+1)$.
10. $(a-12)(a-1)$.
11. $(a+12)(a-1)$.
12. $(a-12)(a+1)$.
13. $(a+6)(a+2)$.
14. $(a-6)(a-2)$.
15. $(a+6)(a-2)$.
16. $(a-6)(a+2)$.
17. $(a+4)(a+3)$.
18. $(a-4)(a-3)$.
19. $(a-4)(a+3)$.
20. $(a+4)(a-3)$.
21. $(x+y)(x-y)$.
22. $(x+2y)(x-2y)$.
23. $(2x+y)(2x-y)$.
24. $(x+1)(x-1)$.
25. $(xy+1)(xy-1)$.
26. $(2a+3b)(2a-3b)$.
27. $(x+y-1)(x-y+1)$.
28. $(x+y+1)(x-y-1)$.
29. $(3ax+2by)(3ax-2by)$.
30. $(2a)(2x)$.
31. $(a-x+b+y)(a-x-b-y)$.
32. $(a+x+b-y)(a+x-b+y)$.
33. $(x+y+1)(x-y+5)$.
34. $(x+y+1)(x-y-5)$.
35. $(2x+3y+1)(2x-3y+5)$.
36. $(x^2+y^2+xy)(x^2+y^2-xy)$.

37. $(x^2-y^2+xy)(x^2-y^2-xy)$. **38.** $(x-4y)(x+2y)$.
39. $(3x+2y)(2x-3y)$. **40.** $3(2x-y)(x+2y)$.
41. $(4x+3y)(3x-4y)$. **42.** $(b-c)(b-c+3)(b-c-3)$.
43. $2(2x-y)(x+2y)$. **44.** $(a-b)(a+b+c)$.
45. $(a+b)(b+c)(c+a)$. **46.** $3x(2a-x)(2a^2-2ax+5x^2)$.

Ex. XII (c) Page 129

3. $(x-1)(x-2)(x+3)(x+4)$. **4.** $(a+1)(a-2)(a-3)(a-4)$.

7. $ab(a-b)+bc(b-c)+ca(c-a)$, $a^2(b-c)+b^2(c-a)+c^2(a-b)$, $(a-b)+(b-c)+(c-d)+(d-a)$, $ab(b-c)+bc(c-d)+cd(d-a)+da(a-b)$, $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}$.

Ex. XII (d) Page 129

1. 3, 2. **2.** -3, -2. **3.** $\frac{3}{2}$, 2. **4.** 4, -3. **5.** 6, -2.
6. 1, -15. **7.** 8, 4. **8.** -5, $\frac{1}{3}$. **9.** 2, ± 3. **10.** ± 2.
11. 3, -7. **12.** -2. **13.** 3, $\frac{2}{3}$, $-\frac{3}{2}$. **14.** $2a$, a. **15.** a, $-2a$.
16. $-\frac{2}{a}$, $-\frac{1}{a}$. **17.** $(a-1)$, $-(a+1)$. **18.** $\frac{3}{4}$, $\frac{2}{3}$. **19.** a, b. **20.** 2.

Ex. XII (e) Page 133

1. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$. **2.** $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$.
3. $(x-y)(x^2+xy+y^2)$. **4.** $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$.
5. $(x+y)(x-y)(x^2+y^2)$. **6.** $(x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6)$.
7. $(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6)$. **8.** $x^3-x^2y+xy^2-y^3$.
9. $x^3+x^2y+xy^2+y^3$. **10.** $8x^3-12x^2y+18xy^2-27y^3$.
11. $4x^2+6xy+9y^2$. **12.** $x^6+x^3y^3+y^6$. **13.** $x^4+x^2y^2+y^4$.
14. x^3+y^3. **15.** $a^4+2a^3b+4a^2b^2+8ab^3+16b^4$.
16. $(x-4)(x+1)$. **17.** $(\mathrm{R}+r)(\mathrm{R}^2-\mathrm{R}r+r^2)$, $(\mathrm{R}-r)(\mathrm{R}^2+\mathrm{R}r+r^2)$.
18. $(a+b)^2+4(a+b)(c-d)+16(c-d)^2$.
19. $(a-b)^3-(a-b)^2(b-c)+(a-b)(b-c)^2-(b-c)^3$.
20. $a^2+3b^2+4c^2+2ac-6bc$.

Ex. XII (f) Page 135

1. $(x-4)$, $(x-4)(x^2-1)$. **2.** $(x+3)$, $x(x+3)(x-4)(x-1)$.
3. 2, $4x^2(x^2-2x-3)$. **4.** (x^2-4), $(x+1)(x^3-x^2-4x+4)$.
5. $(3a+2)$, $(3a+2)(2a-1)(a+1)(a^2-a+1)$.
6. $(a-b)$, $(a-b)^2(a+b)(a-2b)$. **7.** 1, a^6-b^6. **9.** $\frac{x+1}{x-1}$.

10. $\dfrac{2x-3a}{4x^2-6ax+9a^2}$. 11. $\dfrac{2x^2-1}{4x^2+3}$. 12. $\dfrac{(x-4)(x-7)}{x^2}$

13. $\dfrac{(x-1)(x-2)}{x^2}$. 14. $\dfrac{x-a}{(a-b)(c-a)}$. 15. $-\dfrac{2}{x^2-4}$.

16. $\dfrac{a}{1-a}$. 17. $\dfrac{x}{(x-y)^2}$. 18. a^2-1. 19. $\dfrac{1}{a}+\dfrac{1}{b}, \dfrac{2}{a+b}$.

20. $\dfrac{4x^2-15x+14}{1-x^2}$. 21. $\dfrac{bx}{a-b}$. 22. (i) $\dfrac{x+3y}{x^2-y^2}$, (ii) $\dfrac{2}{x^2+5x+6}$.

23. (i) $\dfrac{(c-a)(a^2+ab+b^2)+c^3}{(b-c)(c-a)}$, (ii) 0. 24. 1. 25. $\dfrac{a^2+b^2}{a^4+b^4}$.

26. $-\dfrac{y^2}{(x+y)^2}$. 28. 1. 29. -1. 30. $(a+b)$.

Ex. XIII (A) Page 137

1·41421, 1·73205, 2·23607, 2·64575, 2·82843, 3·16228, 3·31662, 3·46410, 3·60555.

Ex. XIII (B) Page 138

2. $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{5}$, $6\sqrt{7}$. 3. 2·828, 5·196, 6·928, 3·464, 6·708.

4. (i) $9\sqrt{2}-7\sqrt{3}$, (ii) $5\sqrt{3}-4\sqrt{5}$. 5. (i) 60, (ii) $\sqrt{30}$, (iii) $3+\sqrt{6}$, (iv) 12.

6. $11\sqrt{6}-2$. 7. $a+2\sqrt{ab}+b$, $a-2\sqrt{ab}+b$. 8. (i) $3(\sqrt{5}+\sqrt{2})$, (ii) $5\sqrt{3}$.

9. $11\sqrt{5}$. 10. (i) $2\sqrt{3}$, (ii) $\sqrt{3}-\sqrt{2}-1$ or $1+\sqrt{2}-\sqrt{3}$.

11. 7·978. 12. (i) 3·069, 0, (ii) ·1414, ·447, ·3146, 1·201.

13. $\dfrac{14+4\sqrt{6}}{11-6\sqrt{2}}$. 14. $(x+\sqrt{3})(x-\sqrt{3})$, $(x\sqrt{2}+\sqrt{3})(x\sqrt{2}-\sqrt{3})$.

Ex. XIII (C) Page 140

1. ·4472, ·577, 2·268, ·816. 2. 2·414, ·414, ·586. 3. ·892, 5·828.

4. $1\frac{17}{22}$. 5. 0. 6. (i) -6, (ii) $3a^2-2b^2$, (iii) $-3\frac{1}{8}$. 7. $8\sqrt{3}$.

8. (i) $3+2\sqrt{2}=5{\cdot}828$, (ii) $8+3\sqrt{6}=15{\cdot}35$, (iii) $\dfrac{24-\sqrt{15}}{33}={\cdot}61$,
(iv) $\dfrac{2\sqrt{5}+3\sqrt{10}+2\sqrt{6}+4}{11}=2{\cdot}376$.

9. $-5\sqrt{2}$. 10. 5. 11. ·173. 12. $3\sqrt{2}$.

Ex. XIII (D) Page 142

1. $\sqrt{2}:1:1$. 3. 10, $5\sqrt{3}$, 5. 4. 800 sq. yds., $20\sqrt{2}$ yds.

6. $·293a$, $·0537a^2$. 7. $·289x$, $·0722x^2$. 8. $·155x$, $·0269x^2$.

9. $5\sqrt{2}$ ins. 10. $\dfrac{5\sqrt{3}}{4}$ ins. 11. 24 sq. ins.

Ex. XIV (A) Page 147

1. 1·4, 2·1, 1·2. **2.** 1·6, 3·6, −0·2. **3.** 1, 1·5.
4. 0·1, 2·1, 3·1. **5.** 0·5, 1·1, 0·2, 0·6, 2·8. **6.** 1·4, 2·4, 4·4.

Ex. XIV (B) Page 149

1. $a^{\frac{5}{3}}$, $a^{\frac{7}{2}}$, $a^{\frac{1}{2}}$, $a^{\frac{1}{5}}$, $(a+b)^{\frac{3}{2}}$. **2.** $a-a^{\frac{1}{3}}b^{\frac{1}{4}}+a^{\frac{2}{3}}b^{\frac{1}{2}}-b^{\frac{3}{4}}$. **3.** $a^{\frac{1}{4}}-b^{\frac{1}{4}}$.
4. 0·4, 0·27, 0·1. **5.** 0·45, 0·2, 0·47.
6. 0·78, 0·18, 1·18, 0·9, 0·96, 1·44, 0·36, 0·7, 0·52, 1·52.

Ex. XIV (C) Page 151

1. $\bar{2}$, 4, 3, $\bar{1}$, $\bar{3}$. **2.** Nil, 3, 1, 5, 2, 7. **3.** 2·4915, $\bar{1}$·016, $\bar{1}$·2005, 0·299.
4. 1·48, 2·52, 4·234, 0·184. **5.** 16·21, 14·34. **6.** $\bar{1}$·3684, $\bar{2}$·3, $\bar{1}$·87.

Ex. XIV (D) Page 153

1. 0·2175, 1·2175, 2·2175, 3·2175, $\bar{1}$·2175, $\bar{2}$·2175, $\bar{3}$·2175.
2. 0·4771, 1·4771, 2·4771, $\bar{1}$·4771, 2·5159, 1·5159, $\bar{3}$·5159.
3. 1·9523, 3·9523, 1·9523, 3·7042, 0·6995, 0·4972.
4. 0, $\bar{1}$, $\bar{3}$, 1·0004, 2·0009, 3·0009.

Ex. XIV (E) Page 155

1. 1·473, 147·3, 0·01473, 0·1473. **2.** 36·92, 0·3692, 3692.
3. $\bar{1}$·2687, 0·1852. **4.** 1. **5.** $\bar{3}$·6358, 0·004323.
6. 0·05495. **7.** 0·1658, 0·02607. **8.** 0·1452.

Ex. XIV (F) Page 157

1. 0·5933. **2.** 43·07. **3.** 106,900. **4.** 1·455. **5.** 177·4.
6. 3·729. **7.** 8·206. **8.** 27·49, 0·2749. **9.** 1·877. **10.** 2·975.
11. (i) 4·365, (ii) 0·6374, (iii) 1·315. **12.** 4·232. **13.** 4341.
14. 3·634. **15.** 0·419. **16.** 4·42. **17.** 4·767. **18.** 83·685.
19. 3·1827. **20.** $\bar{1}$·875.

Ex. XV Page 166

1. $2x^2-3x-4$. **2.** $-x^2+6x-3$. **3.** $2x^2-3x+1$. **4.** x^2-6.
5. z^2+2. **8.** x^2+2x+3. **9.** 3, $-\frac{1}{2}$. **11.** $W=0{\cdot}785d^2$.
12. Ft. per sec. 48, 80, 113·1, 143; ft. 25, 100, $156\frac{1}{4}$, 289.
14. $z=x+1$. **15.** 0. **18.** The values of y are equal at $x=\pm 2$.
19. (i) $y=0{\cdot}0165x^2+0{\cdot}635x+13$, (ii) $y=0{\cdot}005x^2+0{\cdot}065x+2{\cdot}5$.

Ex. XVI (A) Page 170

1. $3, -1$. **2.** $2, -4$. **3.** $0, -2$. **4.** 3. **5.** 4, roots are equal.

6. $-11, -10$. **7.** $-\frac{3}{2}, -\frac{2}{3}$. **8.** $1, -\frac{1}{6}$. **9.** $1, -3$. **10.** $\frac{4}{3}, \frac{3}{2}$.

11. $\frac{7}{5}, -\frac{5}{2}$. **12.** $1, -\frac{7}{4}$. **13.** $1, -\frac{3}{2}$. **14.** $-\frac{1}{8}, -\frac{3}{2}$.

15. $1{\cdot}66, -{\cdot}361$. **16.** $2 \pm 2\sqrt{3}$. **17.** $-4\frac{1}{2}, -1$. **18.** $4, -\frac{9}{4}$.

19. $\frac{-5 \pm \sqrt{15}}{2}$. **20.** $\frac{1}{2}, -\frac{1}{4}$. **21.** $\frac{2 \pm 2\sqrt{2}}{3}$.

22. $1{\cdot}34, -1{\cdot}94$. **23.** $5, -3$. **24.** $1, \frac{1}{2}$. **25.** $\frac{1}{2}, -\frac{2}{9}$.

Ex. XVI (B) Page 177

1. $4\sqrt{5}$ cms. **2.** $2\frac{1}{12}$ cms.

3. 96°, 148°; 100·8, 12·3 sq. cms.; 9·2, 4·44 sq. cms.

4. 3 ± 2 ins. **5.** 2·77. **7.** 174 miles. **9.** 123 miles.

Ex. XVII (A) Page 181

14. Parabolic, (i) 2 miles, (ii) 12 miles. **15.** (i) 10 miles, (ii) 1 mile.

Ex. XVII (B) Page 184

2. $\frac{5}{3}x^2 - 10x + 21,\ 3 \pm 3\sqrt{-\frac{2}{5}}$. **3.** $3x^2 - 18x - 48$. **4.** For $x > \frac{\sqrt{2}}{2}$.

6. $-\frac{4ab}{c}\sqrt{b^2 - 4ac},\ x^2 + 4x + 16 = 0$. **7.** $27x^2 - 34x + 3 = 0$.

Ex. XVII (C) Page 185

1. $\pm 4, \pm 2$. **2.** $\pm 3, \pm 2$. **3.** $\pm\sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$. **4.** $2, \sqrt[3]{6}$.

5. $4, 3$. **6.** $-2a, a$. **7.** Once, near $-1{\cdot}52$; two.

Ex. XVII (D) Page 188

2. $x = 4, -3\frac{4}{7}$. $y = 1, -2\frac{1}{35}$. **3.** $x = 7, -4$. $y = 4, -7$. **4.** $x = 0, 6$. $y = 0, 5$. **5.** $x = 6, 3$. $y = \frac{2}{3}, \frac{1}{3}$.

6. $x = 4, 2, -2 \pm \sqrt{6}$. $y = 2, 4, -2 \mp \sqrt{6}$. **7.** $x = \pm 9$. $y = \pm 4$. **14.** $x = 1{\cdot}175, -0{\cdot}425$. $y = -0{\cdot}65, -3{\cdot}85$.

15. $x = \frac{3 \mp \sqrt{6}}{2}$. $y = \frac{\pm\sqrt{6} - 1}{2}$. **16.** (i) $x = 5, 4$; $y = 4, 5$; (ii) $x = \pm 3$. $y = \mp 2$. **17.** $x = 3{\cdot}25, 1{\cdot}92$. $y = 1{\cdot}92, 3{\cdot}25$.

18. (i) $\sqrt{7} + \sqrt{2}$, (ii) $\sqrt{7} - \sqrt{2}$. **19.** $3\sqrt{2} - \sqrt{5}$. **20.** $\sqrt{3} - \sqrt{2}$.

21. $\sqrt{19} + \sqrt{17}$. **22.** $\sqrt{a+b} + \sqrt{a-b}$. **23.** $\sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a}{b}}$.

24. 9 and 5. **25.** 14 and 6. **26.** 13 ins.
27. $5(2+\sqrt{6})$. **28.** 1260 ft. (approx.). **29.** $12\frac{1}{2}$ and $22\frac{1}{2}$ m.p.h.
30. (i) -2 to $+2$, (ii) $2, -2$, (iii) 2 to ∞ and -2 to $-\infty$.

Revision Ex. II Page 190

1. Al. : copper $=19:2$. **2.** $y=5-2x$. **3.** $y=\frac{\sqrt{3}x}{3}-3$. **4.** 53, 22. **5.** 10.

6. (1) $(x+2)(x-1)$; (2) (i) $(x^2+2y^2)(x^2-8y^2)$, (ii) $(a+b)^3(a-b)$, (iii) $(x+1)(x-1)(2x-3y)$, (iv) $2(x-2)(x^3+2x^2+12)$.

7. -14. **8.** (i) $\sqrt{2}(4a-3)+\sqrt{3}(3b+2)$, $-5{\cdot}504$; (ii) $(x+y)(x^2+xy+y^2)$.
11. 3586·07. **12.** $1, 2, -3$. **14.** $7+3x-2x^2$, $(\frac{3}{4}, 8\frac{1}{8})$.

Ex. XVIII (A) Page 195

1. $d=\frac{1}{2}at^2$. **2.** $d=ut-\frac{1}{2}at^2$. **3.** 16 ft., 576 ft., 1152 ft.

5. $h=\frac{1}{2}vt$, $t=\frac{v}{32}$. **6.** 18,000 ft.-lbs., 4500 ft.-lbs., 13,500 ft.-lbs.

7. (i) $w=5i$, (ii) 30 ft.-lbs., (iii) $w=5(l-24)$. **8.** $y=425x+14$ (approx.).

Ex. XVIII (B) Page 208

1. $\frac{2}{3}$. **2.** 3·464 cms. **3.** 4·612 cms. **8.** 31·27 ins., 2·94 c. ft.
9. 48π c. ins., 80π c. ins. **11.** 40 million sq. miles, 52 million sq. miles.
12. $E=13{\cdot}42x-0{\cdot}538x^2$ (approx.). **13.** $w=0{\cdot}00026s^2+0{\cdot}17s$ (approx.).

14. $d=\frac{v^2}{64}$. **15.** $v=4e^2$. **16.** $v=5{\cdot}25r^2$. **17.** $\theta=\frac{3}{2}C^2$.

18. $d=16t^2$. **19.** $36{\cdot}9+0{\cdot}17x-0{\cdot}004x^2=$ Alt. (approx.).
20. $W=15980D^2$. **21.** $P=40C-2C^2$, $C=10$. **22.** 7·938 cms.

23. 0·995 ins. **24.** $\frac{13\pi}{6}$ c.cs., 1 cm. **25.** 2·97 ins.

26. (1) $\frac{16}{\pi}$, (3) 256 ft., (4) 657 sq. ft. (approx.).

27. 17·6 ft., 99 sq. ft. approx. **28.** 67029 c. ft., 374 lbs., 2·24 tons.

Ex. XIX (A) Page 214

1. $2, \frac{2}{\sqrt{3}}, \sqrt{3}$; $\sqrt{2}, \sqrt{2}, 1$; $\frac{2}{\sqrt{3}}, 2, \frac{1}{\sqrt{3}}$; $1, \infty, 0$. **3.** 15·18.

5. $\frac{\sqrt{\sec^2 A-1}}{\sec A}$. **6.** $\frac{3\sqrt{13}}{13}, \frac{4\sqrt{41}}{41}$. **9.** (i) $\frac{\sin A}{\sqrt{1-\sin^2 A}}$, (ii) $\frac{\sqrt{1-\cos^2 A}}{\cos A}$.

Ex. XIX (B) Page 223

2. $50\sqrt{3}$, 50 lbs. **3.** 163·84 lbs. **4.** 505·1 ft. per sec.
5. 12·856 sq. ins. **6.** $43 \cdot 58\pi$ sq. ins. **10.** 20·5 miles, 2·96 miles.
11. 64·3° or 89° approx. **12.** From $36\frac{1}{2}$° N. of E. To $36\frac{1}{2}$° N. of E.
13. $y=\frac{2}{3}x-\frac{1}{18}x^2$. **14.** $y=-0 \cdot 000213x^2+3 \cdot 2x$.
15. 4550 ft. per sec.; if $y>x \tan e$.

Ex. XIX (C) Page 226

4. 0·3828, 0·9239, 0·4144. **5.** 0·9659, −0·2588, −3·7321.
6. (i) 45°; (ii) 0°, 90°, 180°; (iii) 0°; (iv) 45°, 135°. **7.** 45°.
8. $2x\sqrt{1-x^2}$, $1-2x^2$, $\sqrt{\frac{1-\sqrt{1-x^2}}{2}}$, $\sqrt{\frac{1+\sqrt{1-x^2}}{2}}$.
9. $4\sin^2 x-4\sin^4 x$. **10.** $2\sin\frac{A}{2}\sqrt{1-\sin^2\frac{A}{2}}$. **11.** $\frac{\sec^2 x}{\cot y-\tan x}$.

Ex. XX (A) Page 228

1. 3, 5, 23. **2.** $2z^2+3z+2$. **3.** $f(y)=4y^2+7y+1$; $\frac{1}{4}$, -2.
4. $\frac{7}{6}$, -3. **5.** $d=ut+\frac{1}{2}at^2$. **6.** $\frac{2}{x+3}+\frac{1}{x+4}$.
7. $\frac{4}{2x+3}-\frac{3}{3x-2}$. **8.** $\frac{3\sqrt{2}}{4}$. **9.** 0·8037, 3.

Ex. XX (B) Page 231

1. $a=\frac{b}{6}$, 0·75. **4.** 7·56, 2·52 tons. **7.** 5·06 ins.
8. $n=3 \cdot 544$, $k=17740$. **9.** 483·8 million miles. **10.** $30\frac{3}{8}$ secs., 11·312 ins.

Ex. XX (C) Page 233

1. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$,
$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$.

3. $a^3-3a^2b+3ab^2-b^3$, $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$,
$a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

4. $a^{10}-5a^8b^2+10a^6b^4-10a^4b^6+5a^2b^8-b^{10}$.

5. $16a^4-96a^3b+216a^2b^2-216ab^3+81b^4$. **10.** 32. **11.** 1·04.

Ex. XX (D) Page 236

1. 270·459 sq. ins. **2.** 113·427 sq. cms., 1703·352 c.cs. **3.** 250·675 c.cs. **4.** 3·03 %. **5.** 363·6 c.cs. **6.** 1·00006, 0·999, 10·007, 1·004, 0·984, 10·12. **7.** 427° C. (approx.). **8.** 974° C. (approx.). **10.** 1·99 %. **12.** 4 %.

Ex. XX (E) Page 238

1. $\frac{1}{2}\pi R^2$, πR^2. **2.** $2\pi R^2$, $4\pi R^2$. **3.** $\frac{4}{3}\pi a^3$. **4.** $\frac{2}{3}\pi r^3$, $\frac{4}{3}\pi r^3$. **5.** 0·8 % (approx.). **6.** 1·1 %. **7.** πRs. **8.** $\frac{1}{3}\pi R^2 h$, $\pi R^2 h$.

Ex. XXI (A) Page 239

1. (i) 20, 23, 26; (ii) -15, -20, -25; (iii) $19a, 23a, 27a$. **2.** $3a$, $-a$, $-5a$, etc.

3. (i) 6, 10, 14, etc.; (ii) 6, -2, -6, etc.; (iii) -6, -2, 2, etc.; (iv) -6, -10, -14, etc.; (v) 1, -3, -7, etc.; (vi) 0, -2, -4, etc.

4. a, $(a+d)$, $(a+2d)$, etc., one less.

Ex. XXI (B) Page 240

1. (i) 486, 1458, 4374; (ii) $\frac{3}{64}$, $-\frac{3}{256}$, $\frac{3}{1024}$; (iii) $48a^5$, $96a^6$, $192a^7$.

2. $3a$, $-6a^2$, $12a^3$, etc.

3. (i) 1, -2, 4, -8, etc.; (ii) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.; (iii) -2, $\frac{2}{3}$, $-\frac{2}{9}$, etc.

4. a, ar, ar^2, etc., one less. **5.** (i) A.P., (ii) G.P., (iii) A.P., (iv) G.P.

Ex. XXI (C) Page 245

1. -79. **2.** 2, 5, 8, 11, 14, 17. **4.** $6\frac{3}{4}$, $10\frac{1}{2}$, $14\frac{1}{4}$. **5.** 2, -1, -4, -7. **6.** 325. **7.** $\frac{1}{2}n(n+1)$. **8.** 400. **9.** 420. **10.** n^2, $n(n+1)$. **11.** 28. **12.** 1220.

Ex. XXI (D) Page 251

1. 4374. **2.** -13122. **6.** -12, 48, -192. **8.** 728, -364. **9.** $11\frac{61}{64}$. **10.** $\frac{11605}{26244}$. **11.** -3072. **13.** $\frac{173}{330}$, $\frac{461}{1998}$. **14.** $\dfrac{1}{1+\delta}$.

Ex. XXI (E) Page 253

1. (i) $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$; (ii) 4, ∞, -4. **3.** 4, 6.

Ex. XXI (F) Page 254

1. £39. 8s. 0d. **3.** £265. 6s. 0d. **4.** 173,600. **5.** 2·022.
6. 2, 3·556, 6·324, 11·25, 20. **7.** 41·67. **8.** $111\frac{1}{9}$ yds.
9. £279,000 approx., £139,500 approx., over 2 million times as much.

Ex. XXI (G) Page 256

1. (i) 1, (ii) -1, (iii) 0·699, (iv) $-0·301$. **2.** (a) (i) 0·0223, (ii) 0·0458, b, 3·9°.
3. G.P., ratio $\frac{2}{5}$. **4.** Equal. **5.** $1\frac{1}{2}$. **9.** After $n=x$.
10. £7. 7s. 0d., £137 ; £5. 18s. 0d., £172 ; £10. 16s. 0d., £95.
12. $\dfrac{a(1-r^n)}{n(1-r)}$. **13.** $a+\dfrac{n-1}{2}d$. **14.** $\dfrac{3}{2}-\dfrac{1\frac{1}{2}}{3^x}, \dfrac{1\frac{1}{2}}{3^x}$.
15. 14·2 yrs. **16.** $S_n=\dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}$, 280483. **17.** $\dfrac{x^n}{n}$, $x<\dfrac{n}{n-1}$.

Revision Ex. III Page 259

1. $1\frac{1}{4}$, $-2\frac{1}{4}$ (approx.). **3.** $\dfrac{\pi s^3}{24}(\sqrt{3}+2)$, $0·704s$.
6. 2·45 secs., 17·6 secs., 5·39 secs. **8.** $\dfrac{3\sqrt{2}-4}{2}$, $(3\sqrt{2}-4)$.
10. 1·264 ins., 2·905 c. ins.

Ex. XXII (A) Page 263

1. $A=1637d^{\frac{3}{2}}$. **2.** $P=481·9\,V^{-1·066}$. **3.** $y=3x^{-2}+4$.
5. (i) 0, (ii) 2, (iii) 4, 2, -2, (iv) 4, 2, 4, $2\sqrt{-1}$. **6.** 2. **7.** 0, 4, -1.
8. 0, $\frac{2}{3}$, -3. **9.** 1·4235. **10.** ± 1, 4. **12.** 2·718.
14. $y=500(14·3)^{-x}$. **15.** $y=21+x$. **16.** 1·512. **18.** $P=·0757\,V^{3·18}$.

Ex. XXII (B) Page 270

4.

	120°	180°	210°	270°	300°	360°	
sin	0·866	0	−0·5	−1	−0·866	0	
cos	−0·5	−1	−0·866	0	0·5	1	
tan	−1·7321	0	0·5774	∞	−1·7321	0	etc.

5. 90°. **10.** Sine curve.
16. (i) $2\sin A\cos B$, (ii) $2\cos A\sin B$, (iii) $2\cos A\cos B$, (iv) $-2\sin A\sin B$,
(v) $\dfrac{2\tan A(1+\tan^2 B)}{1-\tan^2 A\tan^2 B}$, (vi) $\dfrac{2\tan B(1+\tan^2 A)}{1-\tan^2 A\tan^2 B}$.

Ex. XXIII (A) Page 274

2. (i) 2, (ii) 2, (iii) -2, (iv) -2, (v) $\frac{1}{4}$, (vi) $\frac{1}{4}$. **3.** 0.

Ex. XXIII (D) Page 281

2. (i) $-2x$, (ii) $-6x$, (iii) $-4x$, (iv) $-4x$, (v) $4x+5$, (vi) $-4x-5$. **4.** $-\frac{1}{x^2}$.

Ex. XXIII (E) Page 284

2. 45°. **3.** $1\frac{1}{3}$, 2, -4. **5.** $3x^2+c$, $6x+c$, $\frac{3}{2}x^2-2x+c$, $c+at-\frac{1}{2}t^2$.

Ex. XXIII (F) Page 287

3. $-\frac{3}{4}$.

Ex. XXIII (G) Page 288

2. $\int ax^{(n-1)}dx=\frac{1}{n}ax^n+c$. **3.** $9x^2$, $\frac{3}{4}x^4+c$. **4.** $12x^3-15x^2+4x$.

5. $\frac{1}{3}x^3-\frac{3}{2}x^2+2x+c$. **6.** $6x+\frac{3}{2}x^2-\frac{2}{3}x^3+c$. **7.** $\frac{1}{n+1}ax^{(n+1)}+c$.

Ex. XXIII (H) Page 291

1. $\frac{1}{2}$. **2.** 1. **3.** (a) 2, (b) 0, (c) 1, (d) $\frac{\sqrt{2}}{2}$. **4.** 4.

Ex. XXIII (I) Page 292

2. 0, $\pm a$, max. and min. **4.** 90. **5.** Imaginary.

Ex. XXIII (J) Page 294

1. 1. **2.** -2. **3.** $\frac{\sqrt{3}}{4}$. **4.** (a) $\frac{1}{3}$, (b) 0·5176, (c) $2\sqrt{2}$.

Ex. XXIII (K) Page 296

1. (i) 2·303, (ii) 0·693, (iii) -1·61. **5.** e^x.

8. $\int\frac{dx}{x+3}=\log_e(x+3)+c$. **10.** 1·609, 0·917.

Ex. XXIII (L) Page 298

1. 2. **3.** 1, $\frac{1}{12}(8\sqrt{3}-\pi)$. **5.** $\sqrt{3}$, $\sqrt{3}$ 1. **6.** $2\tan x-x+c$.

Ex. XXIV (A) Page 300

1. 1, 5, 6, 10, 15 revs. p. sec. **2.** 45. **3.** 1024 ft. **5.** $1+4\sqrt{2}$, $x=\sqrt{2}$.

6. $\frac{gx^2}{2v^2}\tan^2 e - x\tan e + \left(\frac{gx^2}{2v^2}+y\right)=0$, 15° or 89° approx.

9. $\frac{1}{4}p \times \frac{1}{4}p$. **10.** $l=3$ ft., $r=\frac{3}{\pi}$ ft.

§3. (2) (iii) 5, 2; (iv) 3; (v) $-\frac{100}{x^3}$; (vi) $\frac{m}{x^2}$, *i.e.* the force; (3) (vi) $11\frac{1}{9}$.

§5. (i) 18,310 ft.-lbs., (ii) 102,200 ft.-lbs. **§10.** $D=e^{-kt}$.

§11. $\frac{ar^x}{\log_e r}$. **§14.** $V=kaF\sin ax$.

§15. $d=10\sin 2\pi t$, $v=20\pi\cos 2\pi t$, accel. $=-40\pi^2\sin 2\pi t$,
$V_{\text{max.}}=20\pi$ ins. p. sec.

§16. $3at^2+2bt$, $6at+2b$, $t=0$ or $-\frac{2b}{3a}$.

§19. (i) $\frac{2}{\pi}$, (ii) $\frac{2}{\pi}$, (iii) 0, (iv) $39\frac{1}{2}°$ (approx.). **§20.** 45°. **§21.** $\frac{1}{2}$.

§22. (ii) $\frac{3\pi+8\sqrt{2}+2}{8}r^2$, $\frac{3\pi+8}{4}r^2$, (v) $-\frac{\sin\theta}{1+\cos\theta}$ or $-\tan\frac{\theta}{2}$.

Ex. XXIV (B) Page 317

5. Each 40 million sq. mls., each 52 million sq. mls., 16·7 million sq. mls.

6. $\frac{c^2+t^2}{2t}$, $\frac{\pi t}{3}(3c^2+t^2)$. **8.** $\pi(2Rt-t^2)$.

Ex. XXV (A) Page 322

1. 6, 40320, 60, 120, $\frac{1}{30}$, 5040. **2.** 3, 70, 10, 6, $\frac{4}{3}$, 2. **3.** 120. **4.** ${}_{10}C_3$.

5. ${}_{20}P_3$. **6.** (i) ${}_5P_2$, (ii) 100. **7.** ${}_{52}P_5$ giving ${}_{52}C_5$ different hands, $(4)^5$.

8. 360. **9.** 6!. **10.** 24.

Ex. XXV (B) Page 325

1. 210. **2.** 448. **3.** 2^6. **4.** 3^5, 1. **5.** 10, 5th and 6th, 126.

6. 8th, 5360. **7.** $x^6 \pm 6x^4 + 15x^2 \pm 20 + \frac{15}{x^2} \pm \frac{6}{x^4} + \frac{1}{x^6}$.

8. $a^{10}+5a^8b^2+10a^6b^4+10a^4b^6+$ etc., $a^3+6\sqrt{a^5b}+15a^2b+20\sqrt{a^3b^3}+$ etc.

9. $x^4y^4-4x^4y^2+6x^4-4\frac{x^4}{y^2}+\frac{x^4}{y^2}$. **11.** $a^{-n}-na^{-(n+1)}b+\frac{n(n+1)}{2!}a^{-(n+2)}b^2-$ etc.

14. $a^{-3}+3a^{-4}b+6a^{-5}+$ etc. **16.** $a^{\frac{1}{x}}+\frac{1}{x}a^{\frac{1-x}{x}}b+\frac{1-x}{2!x^2}a^{\frac{1-2x}{x}}b^2+$ etc.

17. $1+x+\frac{n-1}{2!n}x^2+\frac{(n-1)(n-2)}{3!n^2}x^3+$ etc.

Ex. XXV (c) Page 331

3. 1·609. **4.** $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \text{etc.}$, $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \text{etc.}$, $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{etc.}$

7. 1·551. **8.** 0·2865. **9.** 23·14 lbs. **10.** 0 and ∞, -222350.

12. 0·475.

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1. (i) $-n(a-x)^{n-1}$, (ii) $2\sin x\cos x = \sin 2\theta$, (iii) $-2\sin x\cos x = -\sin 2\theta$, (iv) $2\tan x\sec^2 x$.

2. $n\sin^{(n-1)}x\cos x$, $-n\sin x\cos^{(n-1)}x$, $n\tan^{(n-1)}x\sec^2 x$. **3.** $(\frac{1}{8}\pi + \frac{1}{4})$.

4. $\frac{\pi}{2}(2a^2 + b^2)$ **7.** $\theta = 107\frac{1}{2}^\circ$.